For instance, for $k=2$, $l=1$, we find 1.011, for $k=3$, $l=1$, we find 1.002 and so on. It reaches its largest value for $k=\infty$, $l=k-1$, namely, 1.074, while for $l\leqq k-2$, it is smaller than 1.026. Thus we may often neglect it entirely, and write for F, apart from a constant factor

$$F=x\,x_{kl}^{\frac{1}{2}}\,x_{kl}^{*\,-\frac{1}{2}}\{J_{2l+1}(x_{kl})\cos k\pi+N_{2l+1}(x_{kl})\sin k\pi\} \qquad (32b)$$

and instead of (33a)

$$\tan k\pi=-\frac{J_{2l}(x_{kl})J_{l+\frac12}(\tfrac12 x_k^*)-J_{2l+1}(x_{kl})J_{l-\frac12}(\tfrac12 x_k^*)\dfrac{x_k^*}{x_{kl}^*}+\psi J_{2l+1}(x_{kl})J_{l+\frac12}(\tfrac12 x_k^*)}{N_{2l}(x_{kl})J_{l+\frac12}(\tfrac12 x_k^*)-N_{2l+1}(x_{kl})J_{l-\frac12}(\tfrac12 x_k^*)\dfrac{x_k^*}{x_{kl}^*}+\psi N_{2l+1}(x_{kl})J_{l+\frac12}(\tfrac12 x_k^*)}. \qquad (33b)$$

If we apply Formula (33) to the $2p$ state studied by Tibbs[2] we find

$$k=2.004.$$

The approximate formula (21) gives

$$k=2.005.$$

Unfortunately a check with his result does not mean very much here because his value may lie anywhere between 2.00 and 2.041.

It seems to the author that the elimination of purely numerical work in quantum mechanics is very desirable because of the greater generality gained. The present paper makes the cut-off Coulomb field an analytically soluble case except for energy levels which are so close to the cut-off in the potential energy that only an exponential tail is situated beyond the cut-off radius. More precise criteria, together with formulas to replace (21), (27), and (33) will be given in another paper.

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Theory of Static Fields

I. A Phenomenological Attempt to Determine the Proper Field of an Electron

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According to the phenomenological theory of quantum fields, fluctuation phenomena correspond even to the lowest energy state of a system. The average value of field quantities is zero in the case of a vacuum. This value changes, however, if we introduce a particle into the field. The determination of the average value of the field fluctuation leads immediately to the expressions of the proper field of an electron. The calculation becomes considerably simplified by an appropriate use of Dirac's theory.

A. INTRODUCTION

1. Purpose of the Paper

IT is well known that classical theory does not represent, so far, a consistent scheme of physical phenomena, even if we apply it to a simple problem as the problem of the motion of an electron. Even the value of kinetic energy of an electron cannot be obtained from classical physics in a satisfactory way since the value given by the mechanical picture does not agree at all with the value one would expect from the transformation properties of static electromagnetic fields.

The first part of this paper intends to study the proper field of the electron from the point of view of quantum theory, in order to see whether or not quantum theory leads to modifications of the transformation properties of microscopic static fields which are likely to attenuate the difficulties met in classical physics. It answers at the same time the question in which way quantum theory accounts for static fields which are responsible for elementary macroscopic phenomena as electro- and magnetostatics.

The paper will give an account of a number of investigations which have been carried out during the last years by the author and a few of his collaborators and which could not yet be published in detail because of the events in Europe.

2. A Simple Example

In order to have a general view of the properties of the formalism of quantum field theory, we consider first a scalar field, described by a complete set of orthogonal functions in an arbitrary large cube of volume L^3.

$$\varphi_{\mathbf{k}}=e^{i\omega_k t}u_{\mathbf{k}};\quad u_{\mathbf{k}}=e^{i\mathbf{kr}}/L^{\frac{3}{2}}. \tag{2.1}$$

These obey the Klein-Gordon equation

$$\Delta\varphi-\frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2}-\kappa^2\varphi=0. \tag{2.2}$$

The static solution of Eq. (2.2) is given by

$$\varphi=-Ae^{-\kappa r}/r. \tag{2.3}$$

We shall see how quantum field theory accounts for this solution.

According to the formalism of quantum field theory, we have to consider every single Fourier component (2.1) as an oscillator, the energy of which can take only integral multiple values of $h\omega_k/2\pi$. If no particle is present in the field, the Hamiltonian of the field is given by

$$H=\textstyle\sum_{\mathbf{k}}\frac{1}{2}(p_{\mathbf{k}}^2+\omega_k^2 q_{\mathbf{k}}^2-h\omega_k/2\pi), \tag{2.4}$$

where $p_{\mathbf{k}}$ and $q_{\mathbf{k}}$ are canonical variables,

$$p_{\mathbf{k}}q_{\mathbf{k}}-q_{\mathbf{k}}p_{\mathbf{k}}=h/2\pi i. \tag{2.5}$$

Then, the field is represented by the operator

$$\varphi=\sum_k\tfrac{1}{2}(4\pi)^{\frac{1}{2}}c\left\{\left(q_{\mathbf{k}}+\frac{1}{i\omega_k}p_{\mathbf{k}}\right)u_{\mathbf{k}}+\left(q_{\mathbf{k}}-\frac{1}{i\omega_k}p_{\mathbf{k}}\right)u_{\mathbf{k}}^*\right\}. \tag{2.6}$$

It is well known that the Hamiltonian (2.4) describes fluctuating variables $p_{\mathbf{k}}$ and $q_{\mathbf{k}}$, the average values of which are equal to zero. The average value of the field (2.6) in free space is, therefore, equal to zero.

Introducing an infinitely heavy particle with the coordinates $\mathbf{R}$ into the field, we can consider the kinetic energy of the particle to be a constant and the Hamiltonian of the system can be written

$$\begin{aligned}H=\sum_{\mathbf{k}}\Big\{&\tfrac{1}{2}(p_{\mathbf{k}}^2+\omega_k^2q_{\mathbf{k}}^2-h\omega_k/2\pi)\\&+\tfrac{1}{2}(4\pi)^{\frac{1}{2}}c\cdot A\cdot\left[\left(q_{\mathbf{k}}+\frac{1}{i\omega_k}p_{\mathbf{k}}\right)\cdot u_{\mathbf{k}}(R)+\left(q_{\mathbf{k}}-\frac{1}{i\omega_k}p_{\mathbf{k}}\right)\cdot u_{\mathbf{k}}^*(R)\right]\Big\}\\=\textstyle\sum_{\mathbf{k}}&\tfrac{1}{2}\{(p_{\mathbf{k}}+p_{\mathbf{k}}^0)^2+\omega_k^2(q_{\mathbf{k}}+q_{\mathbf{k}}^0)^2-h\omega_k/2\pi-p_{\mathbf{k}}^{02}-\omega_k^2q_{\mathbf{k}}^{02}\}.\end{aligned} \tag{2.7}$$

The constant A plays the role of an interaction constant or "charge" in this theory. The abbreviations $p_{\mathbf{k}}^0$ and $q_{\mathbf{k}}^0$ are determined by

$$\begin{aligned}p_{\mathbf{k}}^0&=(4\pi)^{\frac{1}{2}}c\cdot A\cdot\frac{u_{\mathbf{k}}(\mathbf{R})-u_{\mathbf{k}}^*(\mathbf{R})}{2i\omega_k};\\q_{\mathbf{k}}^0&=(4\pi)^{\frac{1}{2}}c\cdot A\cdot\frac{u_{\mathbf{k}}(\mathbf{R})+u_{\mathbf{k}}^*(\mathbf{R})}{2\omega_k^2}.\end{aligned} \tag{2.8}$$

The Hamiltonian (2.7) represents an oscillator system, the average values of its canonical variables being shifted by the (negative) amounts given by (2.8). Introducing these average values in the expression (2.6), we obtain immediately the average value of the field

$$\bar{\varphi}=-2\pi c^2A\sum_{\mathbf{k}}\left\{\frac{u_{\mathbf{k}}^*(\mathbf{R})u_{\mathbf{k}}}{\omega_k^2}+\frac{u_{\mathbf{k}}(\mathbf{R})u_{\mathbf{k}}^*}{\omega_k^2}\right\}=-A\frac{e^{-\kappa|\mathbf{r}-\mathbf{R}|}}{|\mathbf{r}-\mathbf{R}|} \tag{2.9}$$

in agreement with expression (2.3), since

$$(\omega_k/c)^2=k^2+\kappa^2$$

and

$$\sum_{\mathbf{k}}\frac{4\pi}{k^2+\kappa^2}\frac{e^{\pm i\mathbf{k}(\mathbf{r}-\mathbf{R})}}{L^3}=\frac{e^{-\kappa|\mathbf{r}-\mathbf{R}|}}{|\mathbf{r}-\mathbf{R}|}.$$

The two last terms in (2.7) represent the infinite self-energy of our field, which, however, is without any interest for our purpose.

Replacing our simplified field model by the oscillator system representing the electromagnetic radiation field and the particle by a Dirac electron, the present scheme permits immediate calculation of the proper field of the electron to the approximation in which the electron is sufficiently heavy not to be appreciably influenced by the field.

Before, however, entering into the discussion of the proper field of the electron, we shall develop a mathematical apparatus, which will not only be helpful in the following calculations, but also simplify the considerations which we will have to present in the later parts of this series of investigations.

B. MATHEMATICAL APPARATUS

3. The Solution of Dirac's Equation

We shall write Dirac's equation in the following form:

$$-\frac{1}{c}\frac{\partial\Psi}{\partial t}+\boldsymbol{\alpha}\,\mathrm{grad}\,\Psi+\frac{i\beta}{\Lambda}\Psi=0. \tag{3.1}$$

The constant Λ can be expressed by

$$\Lambda=h/2\pi mc, \tag{3.2}$$

the matrices $\boldsymbol{\alpha}$ and β are supposed to have the usual form.

We shall call a solution of Eq. (3.1) a complete set of orthogonal functions, depending on the variables x, y, z, t, $\boldsymbol{\alpha}$, and β. One verifies easily, that

$$\Psi_{\mathbf{p}}=\frac{p_0+mc-\gamma\mathbf{p}}{\{L^3 2p_0(p_0+mc)\}^{\frac{1}{2}}}\cdot\exp\left[\frac{2\pi i}{h}(\beta p_0ct+\mathbf{pr})\right] \tag{3.3}$$

is a complete solution of Eq. (3.1) if $\mathbf{p}$ denotes an arbitrary vector in space and

$$p_0^2-p^2=m^2c^2, \tag{3.4}$$

$$\gamma=\beta\boldsymbol{\alpha}. \tag{3.5}$$

We shall refer to the form (3.3) of the solution as the *standard form.*

In Table I the solution (3.3) is written explicitly, without the common normalizing factor.

Every column of (3.3) represents a set of four spinors which is generally called a partial solution of Dirac's equation. The four partial solutions included in (3.3) correspond to two sets of plain waves moving in opposite directions. This is an essential feature of the standard solution.

While the asterisk (*) denotes in the usual way the conjugate complex of a quantity, we shall denote by the sign "til" (˜) the conjugate transposed of a matrix. The conjugate transposed of Eq. (3.1) is

$$-\frac{1}{c}\frac{\partial\tilde{\Psi}}{\partial t}+\mathrm{grad}\,\tilde{\Psi}\boldsymbol{\alpha}-\frac{i}{\Lambda}\tilde{\Psi}\beta=0, \tag{3.6}$$

$$\tilde{\Psi}_{\mathbf{p}}=\exp\left[-\frac{2\pi i}{h}(\beta p_0ct+\mathbf{pr})\right]\times\frac{p_0+mc+\gamma\mathbf{p}}{\{L^3 2p_0(p_0+mc)\}^{\frac{1}{2}}}. \tag{3.7}$$

One verifies easily the normalization conditions

$$\int\tilde{\Psi}_{\mathbf{p}'}\Psi_{\mathbf{p}}d\tau=\delta_{\mathbf{p}'\mathbf{p}}, \tag{3.8}$$

$$\tilde{\Psi}_{\mathbf{p}}\Psi_{\mathbf{p}}=\frac{1}{L^3}, \tag{3.9}$$

$$\Psi_{\mathbf{p}}\tilde{\Psi}_{\mathbf{p}}=\frac{1}{L^3}. \tag{3.10}$$

Dirac's matrices describe quadrivalent variables, the four values of which we shall consider to belong to different values of the *character* of the electron. The character includes two bivalent sets of values, $(\alpha\delta)$ and $(\beta\gamma)$, belonging to different *spins*, and two bivalent sets, $(\alpha\beta)$ and $(\gamma\delta)$, belonging to different *signs* of the electron.

An electromagnetic field splits, for an observer in a given system of reference, into two parts: the electric and the magnetic field. In a some-

TABLE I. Solution of Eq. (3.3).

	$\exp\left[\frac{2\pi i}{h}(\mathrm{pr}-p_0ct)\right]$	$\exp\left[\frac{2\pi i}{h}(\mathrm{pr}-p_0ct)\right]$	$\exp\left[\frac{2\pi i}{h}(\mathrm{pr}+p_0ct)\right]$	$\exp\left[\frac{2\pi i}{h}(\mathrm{pr}+p_0ct)\right]$
α	p_0+mc	0	$+p_z$	$+(p_x-ip_y)$
β	0	p_0+mc	$+(p_x+ip_y)$	$-p_z$
γ	$-p_z$	$-(p_x-ip_y)$	p_0+mc	0
δ	$-(p_x+ip_y)$	$+p_z$	0	p_0+mc
	α	β	γ	δ

what analogous way, the character of an electron splits into spin and sign.

The relation (3.8) accounts for the orthogonality *in space*, the relation (3.9) expresses the orthogonality *in character*.

4. Dirac's Matrices

It is well known that Dirac's matrices 1, β, $\boldsymbol{\alpha}$ form a *fundamental system* of matrices. They permit the establishment of a complete *linear basis system* of 16 matrices which permits the expression of any matrix of four rows and columns by a linear expression. We shall denote the linear basis system as follows

$$\begin{array}{ccc} & \beta & \\ 1 & & -\boldsymbol{\alpha} \\ i\boldsymbol{\gamma} & & -\beta\boldsymbol{\sigma} \\ \tau & & -\boldsymbol{\sigma} \\ & i\beta\tau & \end{array} \tag{4.1}$$

with the following notations

$$\boldsymbol{\gamma}=\beta\boldsymbol{\alpha};\quad \boldsymbol{\sigma}=-\frac{i}{2}(\boldsymbol{\alpha}\times\boldsymbol{\alpha});\quad \tau=\tfrac{1}{3}(\boldsymbol{\alpha}\cdot\boldsymbol{\sigma}). \tag{4.2}$$

The matrices (4.1) are of the Hermitian type.

We shall call matrices of the type

$$\begin{array}{cccc} \circ & \circ & & \\ \circ & \circ & & \\ & & \circ & \circ \\ & & \circ & \circ \end{array} \qquad\qquad \begin{array}{cccc} & & \circ & \circ \\ & & \circ & \circ \\ \circ & \circ & & \\ \circ & \circ & & \end{array}$$

respectively *matrices of first* and *second kind.* The matrices

$$1,\quad \beta,\quad \boldsymbol{\sigma},\quad \beta\boldsymbol{\sigma}$$

are of first kind;

$$\boldsymbol{\alpha},\quad i\boldsymbol{\gamma},\quad \tau,\quad i\beta\tau$$

of second kind.

In general, a matrix will contain parts of both kinds. We shall denote by

$$\begin{aligned} {}^1M&=\tfrac{1}{2}(M+\beta M\beta),\\ {}^2M&=\tfrac{1}{2}(M-\beta M\beta) \end{aligned} \tag{4.3}$$

the part of first and of second kind of the matrix M.[1]

5. The Cartan Transformation[2]

We introduce in Dirac's equation a general interaction term of Hermitian type, $U=\tilde{U}$

$$\frac{1}{c}\frac{\partial\Psi}{\partial t}+\boldsymbol{\alpha}\,\mathrm{grad}\,\Psi+\frac{i}{\Lambda}\beta\Psi+\frac{2\pi i}{h}U\Psi=0. \tag{5.1}$$

The conjugate complex equation to (5.1) is

$$\frac{1}{c}\frac{\partial\psi^*}{\partial t}+\boldsymbol{\alpha}^*\,\mathrm{grad}\,\psi^*-\frac{i}{\Lambda}\beta\Psi^*-\frac{2\pi i}{h}U^*\Psi^*=0. \tag{5.2}$$

Introducing Cartan's matrix

$$\rho=-i\beta\alpha_y=\begin{pmatrix} & & & +1\\ & & -1 & \\ & -1 & & \\ +1 & & & \end{pmatrix},\quad \rho^2=1, \tag{5.3}$$

multiplying (5.2) from both sides by ρ and putting

$$\Phi=\rho\Psi^*\rho, \tag{5.4}$$

we obtain, because

$$\rho\beta\rho=-\beta,\quad \rho\boldsymbol{\alpha}^*\rho=\boldsymbol{\alpha},$$

---

[1] A more detailed investigation on the properties of Dirac's matrices will be found in Fernandes de Sa's thesis, Pôrto 1943, in which the general properties of the microscopic space-time continuum are discussed. We shall have to refer frequently to Fernandes de Sa's paper.

[2] The author is indebted to Mr. Jean Pirenne, Lyon, for having drawn his attention to this important transformation, given by E. Cartan.

the equation

$$-\frac{1}{c}\frac{\partial\Phi}{\partial t}+\boldsymbol{\alpha}\,\mathrm{grad}\,\Phi+\frac{i}{\Lambda}\beta\Phi-\frac{2\pi i}{h}\rho U^{*}\rho\Phi=0. \quad (5.5)$$

In the general case, we have to put

$$\begin{aligned} U&=V+W,\\ V&=A_0+\boldsymbol{\alpha}\mathbf{A}+i\gamma\mathbf{E}-\beta\boldsymbol{\sigma}\mathbf{H}, \quad (5.6)\\ W&=\beta B+\tau C_0-\boldsymbol{\sigma}\mathbf{C}+i\beta\tau D, \end{aligned}$$

where the A, B, C, D, E, and H's are real quantities, forming appropriately chosen tensors of a Lorentz transformation. Since

$$\rho V^{*}\rho=V;\quad \rho W^{*}\rho=-W$$

Eq. (5.5) takes the final form

$$-\frac{1}{c}\frac{\partial\Phi}{\partial t}+\boldsymbol{\alpha}\,\mathrm{grad}\,\Phi+\frac{i}{\Lambda}\beta\Phi-\frac{2\pi i}{h}(V-W)\Phi=0. \quad (5.7)$$

Cartan's transformation provides us with a convenient means to study, with the help of Eq. (5.7) the behavior of states, which, according to the usually considered Eq. (5.1), belong to negative energy values. In the restricted case, in which only an electromagnetic field, represented by the four vector A_0, $\mathbf{A}$ is considered, we have to postulate that no additional assumption be introduced, which destroys the necessary symmetry corresponding to Cartan's transformation (Dirac's "filling up rule").

6. Spatial Rotation and Lorentz Transformation

Since we will not have to use these transformations explicitly, only a brief survey shall be given here. The general theory is contained in Fernandes de Sa's quoted paper.

A spatial rotation can be represented in this formalism by a left-hand operation

$$\Psi'=R\Psi. \quad (6.1)$$

The transformation matrix R is of first kind; it depends on Eulers angles φ, ϑ, ψ and on the matrices 1 and $\boldsymbol{\sigma}$, $R(\varphi, \vartheta, \psi; 1, \boldsymbol{\sigma})$.

An essential new feature appears as soon as we consider Lorentz transformations,

$$\Psi'=L_s\Psi, \quad (6.2)$$

where

$$L_s=Ch\frac{u}{2}-\frac{\mathbf{v}}{v}\boldsymbol{\alpha}Sh\frac{u}{2}, \quad (6.3)$$

$$Ch\frac{u}{2}=\frac{1+\left(1-\frac{v^2}{c^2}\right)^{\frac{1}{2}}}{\left\{2\left(1-\frac{v^2}{c^2}\right)^{\frac{1}{2}}\left[1+\left(1-\frac{v^2}{c^2}\right)^{\frac{1}{2}}\right]\right\}^{\frac{1}{2}}};$$

$$Sh\frac{u}{2}=\frac{v/c}{\left\{2\left(1-\frac{v^2}{c^2}\right)^{\frac{1}{2}}\left[1+\left(1-\frac{v^2}{c^2}\right)^{\frac{1}{2}}\right]\right\}^{\frac{1}{2}}},$$

represents an ordinary spinor transformation.

We have to note that the spinor transformation (6.2) destroys the standard form of our solution. We have, therefore, after a spinor transformation, to rearrange our solution by a unitary transformation in order to reestablish the standard form. Physically, this fact means that the transformation properties of a fluctuating quantity of second kind does not obey the ordinary transformation laws under a Lorentz transformation.

7. General Motion

We shall call a general motion a complete set of orthogonal functions

$$\Phi_i=\Psi_{\mathbf{P}}s_{\mathbf{P}i};\quad s_{\mathbf{P}i}(t)=\int\tilde{\Psi}_{\mathbf{P}}\Phi_i d\tau. \quad (7.1)$$

The transformation coefficients are matrices of four rows and columns, and depend explicitly on time and form a unitary transformation. If we decide to carry out summation on double indices without mentioning explicitly the sign of summation, they satisfy the conditions

$$\tilde{s}_{i\mathbf{p}}\cdot s_{\mathbf{p}k}=\delta_{ik};\quad s_{\mathbf{p}'i}\cdot\tilde{s}_{i\mathbf{p}}=\delta_{\mathbf{p}'\mathbf{p}}. \quad (7.2)$$

While in (7.1) $\Psi_{\mathbf{P}}$ represents a solution of Dirac's Eq. (3.1), Φ_i does not satisfy this equation. In general, it will be possible, however, to find a function U depending explicitly on time in order to have Φ satisfying an equation of the type of (5.1).

C. THE PROPER FIELD OF AN ELECTRON

8. The Radiation Field

In order to account for the existence of photons, which according to the phenomenological theory has not to be explained but has simply to be taken as a postulate, we introduce the operators

$$\theta_{\mathrm{k}}, \quad N_{\mathrm{k}}=\frac{1}{i}\frac{\partial}{\partial\theta_{\mathrm{k}}} \tag{8.1}$$

operating on eigenfunctions of the type

$$\Gamma=\exp\left[i\sum_k\left(n_{\mathrm{k}}'\theta_{\mathrm{k}}+\frac{i}{2}\ln 2\pi\right)\right], \tag{8.2}$$

where the n_{k}' can have any positive or negative integer values including zero. If we want to interpret the numbers n_{k}' as numbers of photons, we have to exclude all solutions (8.2) in which all the values of n_{k}' do not have the same sign. We shall denote by n_{k} positive integer values including zero.

If we denote by

$$w_k=h\omega_k/2\pi \tag{8.3}$$

the energy of a photon corresponding to a wave vector $\mathbf{k}$, the eigenvalue problem of the radiation field becomes

$$\left(+\frac{\partial}{\partial t}+\frac{2\pi i}{h}\sum_{\mathrm{k}} w_k N_{\mathrm{k}}\right)\Gamma=0. \tag{8.4}$$

We have to attribute to the field energy the same sign as to the energy of the particles we want the field to interact with. We have, therefore, to introduce a scalar operator ϵ which represents the sign of the electron charge, $\epsilon^2=1$. We shall suppose ϵ to commute with Dirac's matrices. It is, for our considerations, an essential fact that Dirac's theory does not contain any operator of the type required here. Phenomenological theory has, therefore, to introduce this operator *ad hoc*, in an unsatisfactory way.

The solution of (8.4) may now be written

$$\Gamma=\exp\left[i\sum_k\left\{\frac{2\pi}{h}\eta_k\omega_k\epsilon t-\epsilon\eta_{\mathrm{k}}\theta_{\mathrm{k}}+\frac{i}{2}\ln 2\pi\right\}\right]. \tag{8.5}$$

The vector potential of the field will be determined below. We note, however, that the radiation field is described in classical theory by a complete set of orthogonal functions

$$\mathbf{A}_{\mathrm{k}}=\frac{hc}{(\pi\omega_k)^{\frac{1}{2}}}\mathbf{A}_{\mathrm{k}}u_{\mathrm{k}}, \tag{8.6}$$

where $\mathbf{A}_{\mathrm{k}}$ is a unit vector perpendicular to $\mathbf{k}$, determining the polarization of the plane wave. u_{k} has already been defined by (2.1). The amplitude in (8.6) is normalized to correspond to the energy of one photon w_k.

9. The Interaction Energy

For the interaction energy we can use the usual expression, but we need to take care that only electrons and photons of the same energy sign interact with each other. The interaction energy becomes

$$\begin{aligned}U(\epsilon)=\sum_{\mathrm{k}}-e\cdot\mathbf{A}_{\mathrm{k}}\cdot\boldsymbol{\alpha}\cdot\frac{hc}{(2\pi\omega_k)^{\frac{1}{2}}}\{&\tfrac{1}{2}(1-\epsilon)\cdot[(N_{\mathrm{k}})^{\frac{1}{2}}\cdot\exp(i\theta_{\mathrm{k}})\cdot u_{\mathrm{k}}^*+\exp(-i\theta_{\mathrm{k}})\cdot(N_{\mathrm{k}})^{\frac{1}{2}}\cdot u_{\mathrm{k}}]\\&+\tfrac{1}{2}(1+\epsilon)\cdot[(N_{\mathrm{k}}^*)^{\frac{1}{2}}\cdot\exp(-i\theta_{\mathrm{k}})\cdot u_{\mathrm{k}}+\exp(i\theta_{\mathrm{k}})\cdot(N_{\mathrm{k}}^*)^{\frac{1}{2}}\cdot u_{\mathrm{k}}^*]\}.\end{aligned} \tag{9.1}$$

The operators $\sqrt{N}$ and $\sqrt{N^*}$ can be expressed by a differentiation of the order $\frac{1}{2}$

$$\sqrt{N}=\frac{1}{\sqrt{i}}D^{\frac{1}{2}};\quad \sqrt{N^*}=\frac{1}{\sqrt{-i}}D^{\frac{1}{2}}, \tag{9.2}$$

which is defined by

$$D^{\frac{1}{2}}e^{n\theta}=\sqrt{n}\cdot e^{n\theta};\quad D^{\frac{1}{2}}\cdot D^{\frac{1}{2}}=\frac{\partial}{\partial\theta}.$$

With the help of (9.2) the interaction energy (9.1) can be written

$$U(\epsilon)=\sum_{\mathbf{k}}-e\cdot\mathbf{A}_{\mathbf{k}}\cdot\boldsymbol{\alpha}\cdot\frac{hc}{(2\pi\omega_k)^{\frac{1}{2}}}\cdot\frac{\exp\left(i\epsilon\frac{\pi}{4}\right)}{L^{\frac{3}{2}}}\{D_{\mathbf{k}}^{\frac{1}{2}}\exp\left[i\epsilon(\mathbf{kr}-\theta_{\mathbf{k}})\right]+\exp\left[-i\epsilon(\mathbf{kr}-\theta_{\mathbf{k}})\right]D_{\mathbf{k}}^{\frac{1}{2}}\} \tag{9.3}$$

and we can consider

$$\mathbf{A}=\sum_{\mathbf{k}}\mathbf{A}_{\mathbf{k}}\frac{hc}{(2\pi\omega_k)^{\frac{1}{2}}}\frac{\exp\left(i\epsilon\frac{\pi}{4}\right)}{L^{\frac{3}{2}}}\{D_{\mathbf{k}}^{\frac{1}{2}}\exp\left[i\epsilon(\mathbf{kr}-\theta_{\mathbf{k}})\right]+\exp\left[-i\epsilon(\mathbf{kr}-\theta_{\mathbf{k}})\right]D_{\mathbf{k}}^{\frac{1}{2}}\} \tag{9.4}$$

to be the vector potential operator of the field.

The wave equation of the complete system becomes

$$\left\{-\frac{\partial}{\partial t}+c\boldsymbol{\alpha}\,\mathrm{grad}+\frac{i\beta}{\Lambda}+\frac{2\pi i}{h}U(\epsilon)-\frac{2\pi i}{h}\sum_{\mathbf{k}}w_k\cdot N_{\mathbf{k}}\right\}\bar{\Psi}=0 \tag{9.5}$$

and transforms under a Cartan transformation to

$$\left\{-\frac{\partial}{\partial t}+c\boldsymbol{\alpha}\,\mathrm{grad}+\frac{i\beta}{\Lambda}-\frac{2\pi i}{h}U(-\epsilon)-\frac{2\pi i}{h}\sum_{\mathbf{k}}w_k\cdot N_{\mathbf{k}}\right\}\bar{\Phi}=0. \tag{9.6}$$

The functions $\bar{\Psi}$ and $\bar{\Phi}$ depend not only on the electron variables x, y, z, t, $\boldsymbol{\alpha}$, β, but also on the field variables ϵ, $\theta_{\mathbf{k}}$. The expressions (9.5) and (9.6) account correctly for the relation between electron and positron, while the usual interaction term, together with Dirac's filling up rule, does not account in a correct way for the proper field of an electron.

It can be objected, however, that our interaction term (9.3) is not in agreement with experiments since it does not contain any rule which permits the exclusion of spontaneous radiation transitions between states of different energy sign. This objection cannot be maintained. The quoted transitions occur only in second order while we shall show here that (9.3) applies only to non-relativistic first-order problems and that not only the interaction term, but even the whole scheme of quantum field theory has to be abandoned. The question of the quoted radiative transitions does, therefore, not appear explicitly in our considerations.

10. The Field Perturbation

We have now to solve, by an appropriate method of approximation, Eq. (9.5), and we must determine the average value of the ground state of the field (9.4). Supposing that no free photons are present in the field and that the electron has the momentum $\mathbf{p}$, we can represent the solution of (9.5) in first approximation by a series which corresponds to states in which one "virtual" photon has been emitted,

$$\bar{\Psi}=\Psi_{\mathbf{p}}\cdot\exp\left(-\sum\tfrac{1}{2}\ln 2\pi\right)+\sum_{\mathbf{p}'\mathbf{k}}\Psi_{\mathbf{p}'}\cdot\exp\left(\frac{2\pi i}{h}\epsilon\omega_k t-i\epsilon\theta_k-\sum\tfrac{1}{2}\ln 2\pi\right)c_{\mathbf{p}'\mathbf{k}}, \tag{10.1}$$

where the coefficient matrices $c_{\mathbf{p}'\mathbf{k}}$ remain to be determined.

Introducing (10.1) into (9.5) we obtain immediately

$$\sum_{\mathbf{p}'\mathbf{k}}\Psi_{\mathbf{p}'}\exp\left(\frac{2\pi i}{h}\epsilon\omega_k t-i\epsilon\theta_{\mathbf{k}}-\sum\tfrac{1}{2}\ln 2\pi\right)\frac{dc_{\mathbf{p}'\mathbf{k}}}{dt}=\frac{2\pi i}{h}U(\epsilon)\Psi_{\mathbf{p}}\exp\left(-\sum\tfrac{1}{2}\ln 2\pi\right)$$

and, with the help of relation (3.8)

$$\frac{dc_{\mathbf{p}'\mathbf{k}}}{dt}=\frac{2\pi i}{h}\frac{1}{2\pi}\int\exp\left(i\epsilon\theta_{\mathbf{k}}-\frac{2\pi i}{h}\epsilon\omega_k t\right)\times\tilde{\psi}_{\mathbf{p}'}U(\epsilon)\psi_{\mathbf{p}}d\tau d\theta_{\mathbf{k}}. \tag{10.2}$$

In order to evaluate the relation (10.2), we remember that we are interested only in small values of momentum

$$\mathbf{p},\quad \mathbf{p}',\quad \frac{h}{2\pi}\mathbf{k} \ll mc. \tag{10.3}$$

The restriction (10.3) is due to the fact that we cannot expect our method to give reliable results as soon as the particle cannot be considered to be very heavy. We shall see later that our restriction limits our calculation of the proper field of the electron to large distances of the particle, $r \gg \Lambda$. For the present calculation, it implies, however, a considerable simplification, since we can put

$$p_0 = p_0' = mc$$

and since we can neglect all terms which are of higher than first order in the momenta indicated in (10.3). One verifies easily that

$$\begin{aligned}\frac{dc_{\mathbf{p}'\mathbf{k}}}{dt} &= -\frac{2\pi i}{h}\frac{ehc}{(2\pi\omega_k)^{\frac{1}{2}}}\frac{\mathbf{A}_\mathbf{k}}{L^{\frac{3}{2}}}\int \exp\left(-\frac{2\pi i}{h}\epsilon\omega_k t\right)\tilde{\Psi}_{\mathbf{p}'}\boldsymbol{\alpha}\Psi_{\mathbf{p}}e^{i\epsilon\mathbf{k}\mathbf{r}}d\tau \\ &= -\frac{2\pi i}{h}\frac{ehc}{(2\pi\omega_k)^{\frac{1}{2}}}\frac{\mathbf{A}_\mathbf{k}}{L^{\frac{3}{2}}}\exp\left[-\frac{2\pi i}{h}(\beta mc^2+\epsilon\omega_k)t\right]\left\{\left(1+\frac{\gamma\mathbf{p}'}{2mc}\right)\boldsymbol{\alpha}\left(1-\frac{\gamma\mathbf{p}}{2mc}\right)\right\}\exp\left(\frac{2\pi i}{h}\beta mc^2 t\right) \\ &= -\frac{2\pi i}{h}\frac{ehc}{(2\pi\omega_k)^{\frac{1}{2}}}\frac{\mathbf{A}_\mathbf{k}}{L^{\frac{3}{2}}}\left\{\boldsymbol{\alpha}\exp\left(\frac{2\pi i}{h}2\beta mc^2 t\right)-\frac{\beta}{2mc}[\mathbf{p}'+\mathbf{p}+i(\boldsymbol{\sigma}\times\mathbf{p}'-\mathbf{p})]\exp\left(-\frac{2\pi i}{h}\epsilon\omega_k t\right)\right\},\end{aligned} \tag{10.4}$$

with

$$\mathbf{p}-\mathbf{p}'+\epsilon\frac{h}{2\pi}\mathbf{k}=0$$

and, finally,

$$c_{\mathbf{p}'\mathbf{k}} = -\frac{ehc}{(2\pi\omega_k L^3)^{\frac{1}{2}}}\left\{\frac{\mathbf{A}_\mathbf{k}\boldsymbol{\alpha}\cdot\beta}{2mc^2}\exp\left[\frac{2\pi i}{h}(2\beta mc^2-\epsilon\omega_k)t\right]-\frac{\beta}{2mc\epsilon\omega_k}\mathbf{A}_\mathbf{k}\left[2\mathbf{p}+i\left(\boldsymbol{\sigma}\times\epsilon\frac{h}{2\pi}\mathbf{k}\right)\right]\exp\left(-\frac{2\pi i}{h}\epsilon\omega_k t\right)\right\}. \tag{10.5}$$

11. The Average Value of the Field

We now have to calculate the average value of the field (9.4),

$$\frac{1}{L^3}\bar{\mathbf{A}} = \int \tilde{\bar{\Psi}}\mathbf{A}\bar{\Psi}d\theta. \tag{11.1}$$

We observe that the field $\mathbf{A}$ depends on coordinates $\mathbf{r}$ in space while the functions $\bar{\Psi}$ depend on the electron coordinates, which in this paragraph shall be denoted by capital letters $\mathbf{R}$. The average field depends, therefore, on time, on the coordinates $\mathbf{r}$ and $\mathbf{R}$, and on the variables ϵ, $\boldsymbol{\alpha}$, and β.

Introducing in (11.1) the respective values from (9.4) and (10.1), we find

$$\frac{1}{L^3}\bar{\mathbf{A}} = \sum_\mathbf{k}\mathbf{A}_\mathbf{k}\frac{hc}{(2\pi\omega_k)^{\frac{1}{2}}}\frac{1}{L^{\frac{3}{2}}}\left\{\tilde{c}_{\mathbf{p}'\mathbf{k}}\tilde{\Psi}_{\mathbf{p}'}\exp\left(i\epsilon\mathbf{k}\mathbf{r}-\frac{2\pi i}{h}\epsilon\omega_k t\right)\psi_\mathbf{p}+\tilde{\Psi}_\mathbf{p}\exp\left(-i\epsilon\mathbf{k}\mathbf{r}+\frac{2\pi i}{h}\epsilon\omega_k t\right)\Psi_{\mathbf{p}'}c_{\mathbf{p}'\mathbf{k}}\right\}.$$

In our approximation, the functions $\bar{\Psi}$ can be replaced, a part from a factor $L^{-\frac{3}{2}}$, directly by the respective exponentials,

$$\bar{\mathbf{A}} = \sum_\mathbf{k}\mathbf{A}_\mathbf{k}\frac{hc}{(2\pi\omega_k)^{\frac{1}{2}}}\frac{1}{L^{\frac{3}{2}}}\left\{\tilde{c}_{\mathbf{p}'\mathbf{k}}\exp\left[-\frac{2\pi i}{h}\epsilon\omega_k t+\frac{2\pi i}{h}(\mathbf{p}-\mathbf{p}')\mathbf{R}+i\epsilon\mathbf{k}\mathbf{r}\right]+\exp\left[\frac{2\pi i}{h}\epsilon\omega_k t-\frac{2\pi i}{h}(\mathbf{p}-\mathbf{p}')\mathbf{R}-i\epsilon\mathbf{k}\mathbf{r}\right]\cdot c_{\mathbf{p}'\mathbf{k}}\right\},$$

where we introduce the values of the coefficients $c_{\mathbf{p}'\mathbf{k}}$ found in (10.5). The result of the calculation is

$$\bar{\mathbf{A}} = \mathbf{A}_c + \mathbf{A}_\sigma + \mathbf{A}_\alpha \tag{11.2}$$

with

$$\mathbf{A}_c=\sum_{\mathbf{k}}+e\frac{h^2c^2}{2\pi\omega_kL^3}\frac{\mathbf{a}_{\mathbf{k}}\mathbf{p}}{mc\epsilon\omega_k}\mathbf{a}_{\mathbf{k}}\beta\{e^{i\epsilon\mathbf{k}(\mathbf{r}-\mathbf{R})}+e^{-i\epsilon\mathbf{k}(\mathbf{r}-\mathbf{R})}\},\tag{11.3}$$

$$\mathbf{A}_\sigma=\sum_{\mathbf{k}}-e\frac{h^2c^2}{2\pi\omega_kL^3}\frac{\beta}{2mc\epsilon\omega_k}\mathbf{A}_{\mathbf{k}}\left[\left(\boldsymbol{\sigma}\times i\epsilon\frac{h}{2\pi}\mathbf{k}\right)\mathbf{A}_{\mathbf{k}}\right]\{e^{i\epsilon\mathbf{k}(\mathbf{r}-\mathbf{R})}-e^{-i\epsilon\mathbf{k}(\mathbf{r}-\mathbf{R})}\}.\tag{11.4}$$

$$\mathbf{A}_\alpha=\sum_{\mathbf{k}}-e\frac{h^2c^2}{2\pi\omega_kL^3}\left\{\exp\left(-\frac{2\pi i}{h}2\beta mc^2t\right)\mathbf{A}_{\mathbf{k}}\beta\frac{\mathbf{A}_{\mathbf{k}}\boldsymbol{\alpha}}{2mc^2}e^{i\epsilon\mathbf{k}(\mathbf{r}-\mathbf{R})}+\mathbf{A}_{\mathbf{k}}\frac{\mathbf{A}_{\mathbf{k}}\boldsymbol{\alpha}}{2mc^2}\beta\exp\left(\frac{2\pi i}{h}2\beta mc^2t\right)e^{-i\epsilon\mathbf{k}(\mathbf{r}-\mathbf{R})}\right\}.\tag{11.5}$$

12. The Summation of the Series

The series (11.3), (11.4), and (11.5) can be evaluated. We note that the summation has to be carried out not only on the wave vector **k**, but also on both directions of polarization. One verifies easily the following relations

$$\sum A_x^2=1-\frac{k_x^2}{k^2};\quad\cdots,\qquad\sum A_xA_y=-\frac{k_xk_y}{k^2};\quad\cdots\tag{12.1}$$

$$\frac{1}{r}=\sum_{\mathbf{k}}\frac{4\pi}{L^3}\frac{1}{k^2}e^{i\mathbf{k}\mathbf{r}},\qquad\frac{1}{r^2}=\sum_{\mathbf{k}}\frac{2\pi^2}{L^3}\frac{1}{k}e^{i\mathbf{k}\mathbf{r}},\qquad\operatorname{grad}\left(\frac{1}{r}\right)=\sum_{\mathbf{k}}\frac{4\pi}{L^3}\frac{i\mathbf{k}}{k^2}e^{i\mathbf{k}\mathbf{r}},\qquad\text{etc.}\tag{12.2}$$

With the help of (12.1) and (12.2), we find immediately

$$\mathbf{A}_c=\beta\epsilon e\left\{\frac{\mathbf{v}}{c}\frac{1}{|\mathbf{r}-\mathbf{R}|}-\tfrac{1}{2}\operatorname{grad}\frac{\mathbf{v}(\mathbf{r}-\mathbf{R})}{c|\mathbf{r}-\mathbf{R}|}\right\},\tag{12.3}$$

$$\mathbf{A}_\sigma=\epsilon\mu\beta\frac{(\boldsymbol{\sigma}\times\mathbf{r}-\mathbf{R})}{|\mathbf{r}-\mathbf{R}|^3},\tag{12.4}$$

$$\mathbf{A}_\alpha=\frac{\mu}{\pi}\left\{\left(\frac{\boldsymbol{\alpha}}{|\mathbf{r}-\mathbf{R}|^2}-\operatorname{grad}\frac{\boldsymbol{\alpha}(\mathbf{r}-\mathbf{R})}{|\mathbf{r}-\mathbf{R}|^2}\right)\beta+\beta\left(\frac{\boldsymbol{\alpha}}{|\mathbf{r}-\mathbf{R}|^2}-\operatorname{grad}\frac{\boldsymbol{\alpha}(\mathbf{r}-\mathbf{R})}{|\mathbf{r}-\mathbf{R}|^2}\right)\right\}\exp\left(\frac{2\pi i}{h}2\beta mc^2t\right),\tag{12.5}$$

with

$$\frac{\mathbf{v}}{c}=\frac{\mathbf{p}}{mc},\qquad\mu=\frac{eh}{4\pi mc}.$$

13. The Proper Field of the Electron

We now have to interpret the expressions (12.3), (12.4), and (12.5). Since the velocity of an electron corresponding to our standard solution (3.3) is equal to $-\beta\mathbf{v}$, $\mathbf{A}_c$ corresponds directly to the magnetic field because of the translation of the Coulomb field

$$A_0=\epsilon\frac{e}{|\mathbf{r}-\mathbf{R}|}.\tag{13.1}$$

The gradient in the vector potential (12.3) does not contribute to the magnetic field and has no physical signification.

$$\mathbf{A}_\sigma=\epsilon\mu\beta\frac{(\boldsymbol{\sigma}\times\mathbf{r}-\mathbf{R})}{|\mathbf{r}-\mathbf{R}|^3}\tag{13.2}$$

represents the vector potential of the magnetic spin field of the electron.

The last field vanishes, since β anticommutes with $\boldsymbol{\alpha}$

$$\mathbf{A}_\alpha=0.\tag{13.3}$$

The proper field of the electron consists, therefore, in our approximation of two parts which belong to the charge and to the magnetic moment of the electron. It obeys Maxwell's equations and accounts correctly for electrostatic and magnetostatic phenomena.

The field $\mathbf{A}_\alpha$ is of second kind. It corresponds to excitations of the field due to virtual pair production. Its average value vanishes. Still, we know that this field plays an important part. Mr. Jean Pirenne has made a thorough investigation about the way in which the field $\mathbf{A}_\alpha$ contributes to the interaction of electrons. He finds, in agreement with (13.3), that no contributions to interaction result at finite distances. Still, the field $\mathbf{A}_\alpha$ gives rise to interaction terms between electrons and positrons at very small distances. These interaction terms are of the order of spin-spin interaction and can be interpreted as forces of the Heisenberg and Majorana type between unlike particles.[3]

14. Thibaud's Experiment

The next task is to investigate the terms of higher order of the electron field. The higher order terms which have been calculated[4] are rather complicated and no longer obey Maxwell's equations. If we want to attribute a physical meaning to them, electrodynamics becomes an open scheme which does not longer permit any theoretical predictions. If the difficulties concerning the transformation properties of static fields are due to deviations of Maxwell's theory, the question arises whether or not Maxwell's theory represents a complete description of the electromagnetic field in vacuum.

Owing to Professor Jean Thibaud's kind interest in the problem, it has been possible to examine this question from the experimental point of view. Let us suppose that a small fraction of the electromagnetic field is not described by a vector potential but by an antisymmetric tensor potential, as some of our quoted calculations indicated. It can be easily shown that such a potential corresponds to the existence of longitudinal photons of very high penetrating power which could, however, be detected if their wave-length approaches Compton's wave-length. Professor Thibaud examined the radiation due to electron impacts under a voltage of about 280 kv. The penetrating power of the radiation has been studied with a thick layer of lead, up to 140 mm. The result was negative; no indications for longitudinal photons were found; the penetrating part of the radiation corresponded exactly to the known cosmic radiation.[5]

We concluded from Thibaud's results that the origin of the fundamental difficulties concerning the transformation properties of static fields is not the electromagnetic part of the problem and has to be searched for elsewhere.

15. Conclusions

The following conclusions can be drawn from the first part of this paper:

1. Elementary electrostatic facts lead to a consistent mathematical scheme which can be applied to Dirac's theory.
2. The phenomenological scheme of quantum field theory is not appropriate to our problem and has to be abandoned.
3. The proper field of the electron depends essentially on the character of the electron.
4. No progress of the theory is possible without a careful study of the kinematical behavior of the electron character according to Dirac's theory.

That study shall be communicated in the second part of this paper.

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[3] Jean Pirenne, Thèse, Lyon (1941). The author has no possibility of knowing at present whether or not this paper has been published in the meantime. See, however, Cahiers de physique **4**, 1, 6 (1941).

[4] One part of these calculations has been made by Mr. S. Höxter.

[5] The author is indebted to Professor Jean Thibaud for the communication of his unpublished results by letter of November 7, 1940 and for his kind permission to refer to his measurements.