Excited States of Nuclear Particles in the Meson-Pair Theory

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A meson-pair theory of nuclear forces is considered in which it is not necessary to use perturbation calculations. Former results in this theory are improved. New considerations are found necessary if the interaction between heavy particle and meson field is large enough to form bound meson states. For such large interactions, discrete stationary states of neutron and proton exist some of which have higher or lower charge and spin than the normal states. The energy required to excite the higher discrete states is of the order of the rest energy of the meson.

THERE is, as yet, no conclusive estimate of the true value of the spin of the meson or of the statistics which the meson obeys. Theoretical work on nuclear forces in which both possibilities for statistics were used, i.e., Einstein-Bose and Fermi-Dirac, has been done on the assumption that nuclear particles can emit and absorb a meson or a pair of mesons and has met with some measure of success in both cases. In these theories it is necessary to cut off the integrations at small lengths which are of the same order of magnitude as the range of nuclear forces. A better understanding of this result which essentially equivocates the existence of a field may perhaps be available if the perturbation method which is usually used can be foregone. It is the further development of a more exact theory of meson-pair (spin $\frac{1}{2}$) forces that this paper has to report.

Previous calculations of nuclear forces in which it was assumed that neutrons and protons can emit and absorb pairs of particles having spin $\frac{1}{2}$ have been made without using the perturbation method. The first of these papers[1] deals only with those cases in which Coulomb energies, the kinetic reaction of neutron and proton and the rest mass of the emitted particles can be neglected safely. That work assumed the field particles to be electrons and positrons and in it is a detailed discussion of the pair production and destruction process involved by the interaction used. The types of interaction which avoid an accumulation of charge about nuclear particles in their lowest states are also found. A second step in this theory[2] takes into account the rest mass of the field particles so that the results can be applied to meson pairs. The validity of the formula obtained is limited, however, to interaction energies which are not strong enough to form discrete stationary states in which one or more mesons are bound to the neutron or proton. The present paper investigates the properties of these discrete states and indicates under what conditions the former calculations must be supplemented.

The meson-pair theory of nuclear forces is based on the assumption that mesons have charge $\pm e$, mass $\sim 1/10$ proton mass and spin $\frac{1}{2}\hbar$ obeying the exclusion principle. Interaction between the field described by the Dirac wave functions belonging to these mesons and a nuclear particle fixed at the origin of coordinates is assumed to be of the form

$$J = -\eta \int d\mathbf{x} \int d\mathbf{x}' \psi(\mathbf{x}')^{+} \beta \psi(\mathbf{x}) u(x)^{*} u(x') \quad (1)$$

in which η is a constant giving the strength of the interaction, $\psi(\mathbf{x})^{+}$, $\psi(\mathbf{x})$ are quantized wave functions of the meson field having four components on which β operates in the usual way and $u(x)$ is an arbitrary spherically symmetrical function replacing the δ-function (which would be required by relativistic invariance). β is the operator which transforms as a scalar. The state of the heavy particle is assumed to be unchanged by this interaction.

By transforming J to momentum space and adding it to the Hamiltonian of free mesons one obtains the Hamiltonian for the mesons in which electromagnetic interactions and the kinetic

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[1] Wigner, Critchfield and Teller, Phys. Rev. **56**, 530 (1939).
[2] C. L. Critchfield and W. E. Lamb, Jr., Phys. Rev. **58**, 46 (1940).

reaction of the nuclear particle are neglected. This Hamiltonian leads, by commutation with the quantized wave function and subsequent separation, to the equation for the wave function of a single meson, $\phi(\mathbf{p})$, of energy E:

$$[E-c(\boldsymbol{\alpha},\ \mathbf{p})-\beta]\phi(\mathbf{p})$$

$$+\eta v(p)\beta\int v(p')^*\phi(\mathbf{p}')dp'=0. \quad (2)$$

The rest energy of the meson, μc^2, is taken as unity. $v(p)$ is the normalized Fourier transform of $u(x)$. Solutions of Eq. (2) which take the rest mass into account have been found[2] for all eigenvalues $|E|>1$. The eigenvalues, E, are determined by the condition that a certain operator on the spin vector have the eigenvalue -1, *viz.*

$$4\pi\beta\eta\int_0^\infty \frac{E+\beta}{E^2-1-c^2p^2}v(p)^2p^2dp=-1, \quad (3)$$

which is considered to be the limit of the sum

$$\frac{4\pi^2\hbar\eta}{L}\sum_{n=1}^{\infty}\frac{\beta E+1}{(E^2-1)-c^2p_n{}^2}v(p_n)^2p_n{}^2=-1, \quad (4)$$

where L is the radius of a sphere in which p_n is considered to be quantized: $p_n=n(\pi\hbar/L)$, and the limit is taken by allowing the radius, L, to become indefinitely large. It is then evident that if $v(p)$ has no zeros, the left-hand side of Eq. (4) as a function of E becomes infinite at every $E^2=1+c^2p_n{}^2$ the sign of the infinity being opposite for opposite directions of approach. The first differential quotient of Eq. (4) with respect to E yields

$$-\frac{4\pi^2\hbar}{L}\eta\beta\sum_{n=1}^{\infty}\frac{c^2p_n{}^2+(E+\beta)^2}{(E^2-1-c^2p_n{}^2)^2}v(p_n)^2p_n{}^2.$$

For a given eigenvalue of β, either 1 or -1, the differential quotient is either positive or negative, respectively, for all values of E. There is then only one solution of Eq. (4) between two successive singularities.

Eigenvalues E satisfying $|E|>1$ can, according to the above arguments, differ only by an infinitesimal amount, $\sim\hbar c/L$, from the values $\pm(1+c^2p_n{}^2)^{\frac{1}{2}}$ and only by summing over all occupied energy levels in the "negative sea" comprising an infinite number of states can the order of the total energy change become of finite or perhaps infinite order. This summation has been presented in explicit form for one heavy particle and implicitly for two heavy particles in reference 2. It is found that the change in $q\equiv(1/c)(E^2-1)^{\frac{1}{2}}$ from its force-free value, p_n, is $x(\pi\hbar/L)$ where

$$\tan\pi x=2\pi^2q(E+\beta)v(q)^2/[-\eta^{-1}-(E+\beta)f(q)],$$

$$f(q)\equiv\frac{4\pi}{c^2}\int_0^\infty[p^2v(p)^2-q^2v(q)^2](q^2-p^2)^{-1}dp.$$

For very small η, $x\rightarrow 0$ as $\eta\rightarrow 0$; this is the case if we take the arctg near zero for small η. The two results for x, obtained with different values of β can be added and the total change in energy in all negative levels for which $|E|>1$ is

$$\Delta E_{\text{con}}=-\frac{2}{\pi}\int_0^\infty\frac{c^2qdq}{(c^2q^2+1)^{\frac{1}{2}}}\operatorname{arctg}\frac{4\pi^2q^2v(q)^2[qf(q)-(c^2\eta q)^{-1}]}{\eta^{-2}-c^2q^2f(q)^2+4\pi^4q^4v(q)^4/c^2+2f(q)/\eta} \quad (5)$$

in which, also, the arctg is to be taken on that branch of the curve which becomes zero as η becomes very small.

In the "hole-theory" which is assumed here the energy of the lowest state (all negative energy levels filled) is negative infinity with or without interaction with nuclear particles. Any finite result which may be obtained from the expression for ΔE_{con} is a finite difference between the infinities in the two cases which is made definite by summing over the same *number* of mesons in each. This procedure is consistent with the conservation of total charge under the interaction.

Although it has been assumed that $v(p)$ has no zeros each exception to this assumption will change the argument for only four states counting

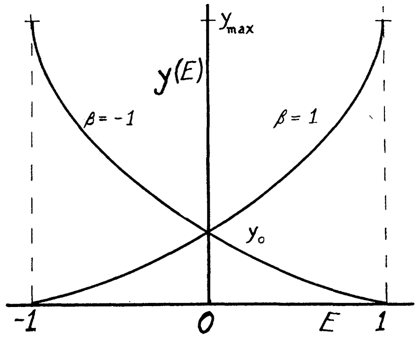

FIG. 1. Sketch of the function $y(E)$. Eigenvalues of E less than one (μc^2) in absolute value are determined by the intersection of this function with a horizontal line at $1/\eta$.

positive and negative energies and both spins and the error made by neglecting this change is infinitesimal. It need be postulated, therefore, that $v(p)$ have only a finite number of zeros in order that the summation over all energy levels obtained before be correct.

The preceding discussion applies to the energy levels for which $|E|>1$. We now consider the possibility of energy levels for which $|E|<1$. For this range of E Eq. (3) presumably has no singularities on the real E axis and there is no need to resort to a summation. Solutions of Eq. (3) are values of E satisfying

$$y(E)=1/\eta \tag{6}$$

$$y(E)\equiv\int_0^\infty \frac{1+\beta E}{c^2p^2+1-E^2}4\pi v(p)^2p^2dp,$$

$$y(E)'=\beta\int_0^\infty \frac{c^2p^2+(E+\beta)^2}{(c^2p^2+1-E^2)^2}4\pi v(p)^2p^2dp.$$

Clearly there is no solution of Eq. (6) for $-1<E<1$ when $\eta<0$. It is, therefore, sufficient to consider only positive values of η. The sign of the derivative of $y(E)$ is the sign of the particular eigenvalue of β that is being considered for the whole range of E. This shows that the extreme values of $y(E)$ are approached as $|E|\rightarrow 1$; beyond these extreme values the eigenvalues of E must lie infinitesimally close to $E=\pm 1$. The maximum values of $y(E)$ are equal and are obtained by taking β and E of the same sign

$$y_{\max}=\frac{8\pi}{c^2}\int_0^\infty v(p)^2dp,$$

which is presumed to be finite. The minimum value of $y(E)$ is then obtained by taking β and E of opposite sign and is seen to be zero. Intersection of the two branches of $y(E)$ corresponding to two eigenvalues of β takes place at $E=0$

$$y(0)=4\pi\int_0^\infty \frac{v(p)^2}{1+c^2p^2}p^2dp.$$

A qualitative sketch of the function $y(E)$ is shown in Fig. 1.

Let E_k be the energy at which the function $v(p)$ has its maximum; then $y_{\max}\sim\mu c^2/E_k^2$. "Discrete states" may then arise if the interaction energy η is of the order of, or larger than, $E_k^2/\mu c^2$. If it be so that the particular choice of η and $v(p)$ is such that η falls in this range $(\eta\gtrsim\sim E_k^2/\mu c^2)$, calculation of the total energy change in the meson field must include the finite energy changes in states between $-\mu c^2$ and μc^2. When spin is taken into account there are four such states, two of negative energy and two of positive energy in general. The lowest state of the system, one heavy particle plus meson field, differs in energy from the lowest state without interaction by

$$\Delta E=\Delta E_{\text{con}}+2(\mu c^2-|E_y|), \tag{7}$$

where E_y is the value of $|E|<1$ which satisfies Eq. (6) and ΔE_{con} is the total energy change in the continuum given by Eq. (5).

The appearance of discrete bound states of a system of particles for sufficiently large interactions is well known from the theory of the deuteron. In the present theory discrete states appear only if $\eta\gtrsim 1/y_{\max}$.

A straightforward extension of the procedure given for one heavy particle can be made for two heavy particles along the line pursued in the previous work. Such a calculation would be important for the forces between the two heavy particles if the interaction energy were of the correct magnitude to give discrete states. The formula for this calculation will not be developed in this paper.

It is the purpose of the present note rather to investigate the states of heavy particle plus meson field which are characterized by the presence of additional mesons or meson pairs. Suppose the interaction between a neutron and the meson field to be that considered above with $y_{\max} > 1/\eta > 0$ so that we have the most general case. Then the lowest state of the system, in which the neutron is considered fixed at a point, is the state in which all negative energy levels of mesons are occupied and all positive levels empty. All levels having $|E| > 1$ belong to the continuum and are changed only infinitesimally by the interaction and in addition there are two levels (of opposite spins) at E_y and two at $-E_y$, $|E_y| < 1$ which are different from μc^2 by a finite amount.

The charge distribution in the lowest state has been proved in an earlier work[1] to be uniform for the interaction that has been used. In higher states of neutron plus meson field the charge distribution is not necessarily uniform. Let us start with the lowest state and create one pair of mesons by removing a meson from the state $-E_1$ and placing it in a state E_2 by supplying the energy E_1+E_2. If E_1 and E_2 (both positive) are larger than μc^2 there will be only an infinitesimal change in charge density, for, these eigenfunctions of the individual mesons differ only infinitesimally from plane waves and give a uniform charge distribution. Consequently, any finite number of pairs produced in which the energy levels involved are not $\pm E_y$ will not produce a change in the neighborhood of the neutron.

Let us now consider the special case that the pair is produced by a transition from $-E_1$ to E_y. The level at E_y is different by a finite amount from the level at μc^2 and will change the charge density in the proximity of the neutron. Departure from a uniform distribution will be entirely the result of the level at E_y being occupied. It is then necessary to calculate $|\psi(\mathbf{x})|^2$ for the state at E_y. According to the former work the eigenfunction belonging to energy E is

$$\phi(\mathbf{p}) = \frac{E+\beta+c(\boldsymbol{\alpha}, \mathbf{p})}{E^2-1-c^2p^2} Cv(p),$$

$$\psi(\mathbf{x}) = (2\pi\hbar)^{-\frac{3}{2}} \int \phi(\mathbf{p}) e^{-ipx} d\mathbf{p},$$

where C is a constant spinor and normalizing factor.

As a special example: If $v(p)$ approach zero more rapidly than p^{-2} as $p \rightarrow \infty$ in all parts of the negative half-plane $\psi(\mathbf{x})$ can be estimated by Cauchy's theorem to contain, in linear combination, $u(x)$ and $\exp[-x(1-E_y^2)^{\frac{1}{2}}/\hbar c]$. The radius of the charge concentration found under this assumption is essentially the larger of the radii of $u(x)^2$ or $\exp[-2r(1-E_y^2)^{\frac{1}{2}}/\hbar c]$. Such a charge accumulation gives the states of the neutron in question the characteristics of discrete states. There are evidently five distinct types of these discrete states and we shall enumerate the lowest levels of each type together with the net charge and possible spins in each level. All negative levels except those having energy $-E_y$ are tacitly assumed to be filled and the number of mesons having energy $\pm E_y$ are given. The meson is assumed to have unit charge. This enumeration appears in Table I.

There is evidently no possibility of interpreting these discrete states so that they may include the proton and neutron as eigenstates of the same system because of the connection between spin and statistics and the fact that protons have unit charge. A rearrangement of levels can be brought about by introducing a spin dependence of potential between meson and nuclear particle; and in this connection it may be observed that if the interaction constant is large enough to form discrete meson states the device[3] previously found necessary to obtain spin dependence of forces in the deuteron may not be required, namely that both a spin-dependent and a spin-independent interaction had to be postulated. Discrete states can be formed only for one eigenvalue of the spin interaction so that the symmetry of the previous calculation (which was

TABLE I. *States of the meson.*

Energy of Excitation	Number in E_y	Number in $-E_y$	Net Charge	Total Spin
zero	0	2	0	$\frac{1}{2}$
E_y	0	1	-1	0, 1
	1	2	1	
$2E_y$	0	0	-2	$\frac{1}{2}$
	1	1	0	$\frac{1}{2}, \frac{3}{2}$
	2	2	2	$\frac{1}{2}$
$3E_y$	1	0	-1	0, 1
	2	1	1	
$4E_y$	2	0	0	$\frac{1}{2}$

[3] C. L. Critchfield, Phys. Rev. **56**, 540 (1939).

for electrons) is destroyed and the possibility of a tensor-tensor interaction alone is open. Furthermore it has been pointed out by Heitler[4] that the existence of discrete states would be very important in the scattering of mesons by nuclei.

In conclusion we may sum up the status of the "exact" solution of the meson-pair theory of nuclear forces as follows: Coulomb interaction and motion of heavy particles are neglected; neutron and proton are assumed to interact with mesons in a particular state $v(p)$ and if the interaction constant η is equal to or greater than

$$\left\{8\pi\mu\int_0^\infty v(p)^2dp\right\}^{-1}$$

discrete states are formed. If η is less than the critical value it is sufficient to consider the effect of the nuclear particle on the continuum of filled meson levels of negative energy.

The author would like to express his thanks to Professor Eugene Paul Wigner for suggesting an investigation of excited states of nuclear particles.

[4] W. Heitler, Nature **145**, 29 (1940).

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Branching Ratios in the Fission of Uranium (235)*

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A survey has been made of the percentages of slow neutron-induced uranium fissions that give rise to the formation of various radioactive series. These percentages were obtained by measuring the number of β-disintegrations of a suitable member of the series after an irradiation under specified geometrical conditions and a quantitative chemical separation. The percentages found for the series thus far investigated vary from about 0.1 percent to 10 percent.

A LARGE number of radioactive series found among the fission products of uranium have been reported. Thus far, however, the work has been confined mostly to the identification and the genetic relationships of the radio-elements which arise from uranium fission. The present work is part of a systematic attempt to determine in a quantitative way the probability that when uranium fission occurs a given radioactive series will appear. We shall call this probability the branching ratio of the radioactive series.

In Table I† are listed those radio-elements which have been found to date with indication, wherever possible, of their genetic relationships and atomic weights. In compiling this table we have relied mainly on the critical survey of Livingood and Seaborg,[1] supplemented by data which have appeared subsequently. In Table I the fission fragments have been arranged in two groups; a light group having atomic weights in the range from 82 to 100 and a heavy group with weights ranging from 127 to 150. To date, 10 radioactive chains have been identified in the light group and 12 in the heavy group. Because of greater analytical ease we have concentrated our effort on the heavy group and have determined the branching ratios of 9 out of 12. The sum of these heavy group fission series amounts to only about 50 percent, (see Table I) indicating that our present knowledge of fission fragments is still incomplete.

The presence of a variety of series may be interpreted by assuming that the original splitting may take place into different fragments. The emission of one or more neutrons further increases the number of possibilities. If no neutrons were emitted the sum of the weights of the two fragments ought to be equal to 236, in the case of fission produced by slow neutrons in U^{235}. If

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† *Note added in proof.*—The 22-hour and the 6.6-hour iodines have now been assigned to atomic weights 133 and 135, respectively, by C. S. Wu, Phys. Rev. **58**, 126 (1940).

[1] J. J. Livingood and G. T. Seaborg, Rev. Mod. Phys. **12**, 30 (1940).