record. It will be noticed that the emission starts from the periphery of the spot and grows inward. Probably the higher temperature of the region adjacent to the nickel plate promoted activation more rapidly than the cooler central portions insulated from the plate by the oxide beneath. The current through the oxide then raised the temperature and caused the emission to spread toward the center. This explanation is the same as that outlined by Myers[9] to account for the hot spots in the emission surfaces.

Prints of the negatives used in comparing emission are shown in Fig. 5. In each picture SrO is the compound on the right, while the compounds on the left are (A) BaO, (B) $Ba(NO_2)_2$, (C) BaS.

[9] L. M. Myers, *Electron Optics, Theoretical and Practical* (Chapman and Hall Ltd., London, 1939), p. 435.

Discussion

The points used to plot the curves of Fig. 3 were obtained from several runs. Since they do not lie above or below the curves a distance of more than 0.25 unit along the ordinate, the error in current density, on a comparative basis, was not greater than the antilog 0.25 or a factor of 1.3.

The absolute values of the current densities appear to be low when compared with the literature.[10, 11] This indicates that the absolute measurement of current density by this method is not as reliable as the comparative measurement.

The authors wish to express their appreciation to Dr. P. H. Carr for his suggestions throughout the course of the work.

[10] Saul Dushman, Rev. Mod. Phys. **2**, 381 (1930).
[11] H. J. Spanner, Ann. d. Physik **75**, 609 (1924).

JANUARY 15, 1941 PHYSICAL REVIEW VOLUME 59

Motion of the Earth's Fluid Core: A Geophysical Problem

D. R. Inglis
The Johns Hopkins University, Baltimore, Maryland

(Received October 30, 1940)

The earth's core may be assumed to have a very low viscosity, such as is characteristic of molten metals. The angular acceleration of the earth is sufficiently large, and the radius of the core is sufficiently great, to raise the question whether the rotation of the central part of the core lags appreciably behind the rotation of the solid mantle. The angular acceleration of the mantle, which is known astronomically, consists of a gradual deceleration and a more pronounced change of direction of the angular velocity, the 27,000-year precession. These two aspects are discussed separately. Three types of force might accelerate the core, viscous force in laminar flow, resistance caused by turbulent flow, and the force of induction associated with the earth's magnetic field. The viscous force is so weak that the interior would be practically unaccelerated if the flow were laminar, and the magnetic induction is expected to be weak enough to permit a large lag between the rotation of the mantle and interior of the core. But the core is so large that the flow should be turbulent. Reasonable assumptions on the nature of the flow in this case, based on empirical data on turbulence near a flat boundary, are used to estimate the lag. It is concluded that the axis of rotation of the interior of the core may be expected to lag behind the axis of the mantle in the precession by an angle of the order of magnitude of a few degrees. Apart from the superposed eddies, points rather near the surface of the core would then move relative to the mantle around horizontal closed paths, approximately a hundred kilometers across, with a period of a day. This would cause a diurnal variation of the earth's magnetism much larger than observed if it were not for the shielding of metallic layers above the core.

Introduction

The fact that transverse seismic waves are apparently not transmitted through the core of the earth[1a] indicates that the matter within the core is almost certainly fluid. The transmission of the longitudinal waves shows[2] that the viscosity is not greater than that of pitch at room temperature, 2×10^{10} g cm sec.$^{-1}$. The viscosity is, however, probably very much

[1] B. Gutenberg, Editor, *Internal Constitution of the Earth* (McGraw-Hill, 1939), (a) Gutenberg, p. 347 and verbal report, Macelwane, p. 277; (b) Jeffries, p. 13.

[2] H. Jeffreys, M. N. R. A. S., Geophys. Supp. **1**, 412 (1926).

less than this upper limit. The core is thought to consist almost entirely of the elements iron and nickel, forming a simple metallic compound which would be expected to have a very definite melting point even at enormous pressure. The plastic state of matter, which reacts as a solid to sudden disturbances but as a fluid to very gradual disturbances, is characteristic of substances consisting of rather complicated molecules, not simple atomic substances like metals. We may, therefore, consider that the most probable order of magnitude of the coefficient of viscosity of the matter in the earth's core is that of molten metals at ordinary pressures, namely, about 10^{-2} g cm sec.$^{-1}$.

It is known that the solid part of the earth outside of the core, which we shall here call the mantle, experiences a rather complicated angular acceleration. This angular acceleration may be separated into two parts, first, a change of the magnitude of the angular velocity, an angular deceleration caused largely by tidal friction, and second, the angular acceleration involved in the 27,000-year precession of the earth's axis about a 24° cone, caused largely by the action of the moon's inhomogeneous gravitational field on the obliquely oriented quadrupole moment of the earth. The question arises whether the rotation of the fluid core keeps up with the rotation of the mantle or whether it lags considerably behind. If it lags behind, there is a rotation of the core relative to the mantle. Since such a relative rotation might have geophysical consequences, including an effect on terrestrial magnetism, we shall here discuss the problem of how much lagging might be expected.

We shall first discuss the effect of the gradual slowing down of the earth's rotation, and leave until later a discussion of the 27,000-year precession. Even though the precession is more important than the deceleration and the two are not actually separable dynamically, it will be helpful first to obtain the orders of magnitude of the various forces in the hypothetical case of deceleration without precession. In order to show that the problem is not entirely trivial, that is, that there might under certain circumstances be a very great lagging effect, we shall first consider the ideal case of laminar flow.

Laminar Flow

The dimensions are so great that this angular deceleration is transmitted to most of the core extremely slowly, if the core is as fluid as molten metal. The deceleration is, of course, transmitted most quickly to the outer layers of the fluid core, and gradually to successively deeper layers. We may obtain an upper limit for the depth of penetration of angular velocity by substituting an initial difference of angular velocity for the gradual deceleration. Since the angular speeds of revolution of the moon and rotation of the earth show that the earth once rotated about six times as fast as it does at present,[1b] we may assume that the initial angular velocity difference is 5Ω, where $\Omega = 7\times10^{-5}$ sec.$^{-1}$ is the present angular speed of the earth.

Anticipating the result that the penetration during the lifetime of the earth is much less than the radius of the core, we may also replace the spherical boundary by a plane solid boundary of a deep fluid. We take this boundary to be the x-y plane, with z directed inward along the radius, and let the velocity have only an x component, u. The equation of motion in the fluid is then

$$\dot{u} = s\partial^2 u/\partial z^2, \tag{1}$$

where $s = \eta/\rho$, the coefficient of viscosity, η, divided by the density, ρ. At $t=0$, we have the initial conditions

$$u=0 \quad \text{for} \quad z=0,$$
$$u=u_0 \quad \text{for} \quad z>0.$$

The problem is identical in form with that of heat conduction into a solid from a hot boundary, and the solution is familiar from the work of Fourier:

$$\begin{aligned}(u_0-u)/u_0 &= (\pi st)^{-\frac{1}{2}}\int_0^\infty e^{-(\lambda+z)^2/4st}d\lambda \\ &= 2\pi^{-\frac{1}{2}}\int_q^\infty e^{-\beta^2}d\beta = 1-\Theta(q),\end{aligned} \tag{2}$$

where $q = z/(4st)^{1/2}$ and $\Theta(q)$ is the probability integral, of which tables are available. Θ is equal to 0.84 if q is unity, and rapidly approaches unity for somewhat larger values of q. The condition for

$$(u_0-u)/u_0 \ll 1$$

is thus that

$$z \gtrless (4st)^{1/2}. \tag{3}$$

Inserting the plausible values $\eta = 10^{-2}$ g cm^{-1} sec.$^{-1}$ and $\rho = 10$ g cm^{-3}, we have $s = 10^{-3}$ cm^2 sec.$^{-1}$. For t we take the age of the earth, say 6×10^{16} sec., and find

$$z \gtrless 1.5 \times 10^7 \text{ cm}. \tag{3'}$$

With these values of the constants, the depth of penetration of the transverse velocity into the fluid is thus much less than the radius of the core $R = 3 \times 10^8$ cm.

Turbulent Flow

One would not, however, expect the flow to be laminar in so large a fluid. In predicting the behavior of the flow, one must be guided by considerations of dynamical similarity. The Reynolds number for this case,

$$uR/s = \omega_r R^2/s$$

is of the order of magnitude 10^{15} or more, if the relative angular speed ω_r is comparable with the rotation of the earth. Experimental data on the flow of fluids past flat plates or through pipes are available up to a Reynolds number of almost 10^{10}. The data on flat plates with an adequately streamlined entering edge indicate a sudden transition from laminar to turbulent flow at about the value 10^5. Without streamlining, the flow is turbulent at even smaller values of the Reynolds number. In the interior of the core, with a Reynolds number of about the order of magnitude 10^{15}, the flow would certainly be expected to be turbulent, unless there should be a stratification of different materials according to density, which would hinder the turbulence.

To make an estimate of the degree of penetration in the extreme case of turbulent flow, we may make use of the empirical relations[3]

$$u(z) = u_0(z/\delta)^{1/7} \quad \text{for} \quad 0 < z < \delta, \tag{4.1}$$

$$\tau_0 = 0.02 \rho u_0^{7/4} (s/\delta)^{1/4}, \tag{4.2}$$

where δ is the depth of penetration of the deceleration from a stationary surface into a deep moving fluid, and τ_0 is the tangential force per unit area on the surface. In analogy with the possibility of an initial relative rotation of the core and mantle of the earth, we first consider the case in which the stationary boundary is introduced at a time $t = 0$ (even though this does not correspond very closely with the methods of obtaining data on turbulence). The change of momentum of the flow caused by the influence of the boundary, per cm^2, is

$$\rho \int_0^\delta (u - u_0) dz = -\rho u_0 \delta / 8.$$

The penetration δ increases with the time, and τ_0 decreases slowly. By equating the rate of change of momentum per cm^2 to τ_0 in (4.2), we obtain

$$\delta^{1/4} d\delta/dt = 0.16 u_0^{3/4} s^{1/4}. \tag{5}$$

After integrating this from zero to T, the age of the earth, we have

$$\delta = 0.3 u_0^{3/5} s^{1/5} T^{4/5} \approx 10^{16} \text{ cm}. \tag{6}$$

This is, of course, very much larger than the dimensions available within the core of the earth. However, even with this enormous depth of penetration, the velocity at a depth less than the radius of the core is still appreciable. According to (4.1) we have

$$u(10^8 \text{ cm}) \approx 10^{-1} u_0. \tag{7}$$

This estimate would not apply to the motion in the earth's core except as a very rough upper limit, because the central part of the core has a comparatively small inertia and can be accelerated to its final equilibrium speed much more easily than can the infinite fluid beyond the region of greatest shear in the case just treated.

Since the failure to treat the inertia properly is probably by far the worst feature of Eq. (7) as applied to the earth, we may make a much better estimate of the motion in the earth's core by improving the treatment of the inertia while retaining the data on turbulence derived from observations of flow past a straight boundary.[3] For this estimate, it will be sufficiently accurate to replace the spherical problem of the earth's core by the problem of an infinite cylinder of radius R

[3] T. v. Kármán, (a) Zeits. f. angew. Math. u. Mechanik **1**, 233 (1921), Eqs. (16) and (17); (b) J. Aero. Sci. **1**, 1 (1934). There Eq. (14), which corresponds to our Eq. (4.2), gives a force about half as great at $R_x = 10^{10}$, and about a third as great at $R_x = 10^{12}$, as his Eqs. (40) and (39) which are more exact for high R_x. We may thus expect the use of the convenient but approximate formula (4.2) to give too large values of the lag of the core, but to lead to results of about the correct order of magnitude, since our final result corresponds to a Reynolds number of about 10^{14}.

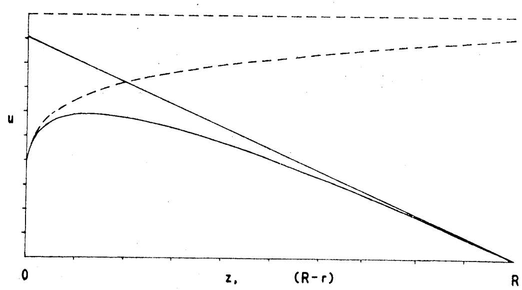

FIG. 1. Average horizontal speed as a function of depth beneath a plane boundary (broken curve) and a cylindrical boundary (solid curve) in the case of turbulent flow. The straight lines indicate initial speeds.

rotating about its axis, since the ratio of the moment of inertia to the area of the surface has the same order of magnitude in both problems. The depth of penetration, δ in Eq. (4) no longer has any meaning. Instead of it, we propose to express the empirical data in terms of the parameter $u(a)$, the velocity at an arbitrarily chosen radius a, considerably less than R. Our initial conditions at $t=0$ become

$$u_0(R)=0,\ u_0(r)=\omega_0 r \quad \text{for} \quad r<R.$$

As an assumption which we shall expect to lead to the correct orders of magnitude only, we shall suppose that the variation of the percentage reduction of velocity with depth in the cylindrical problem is similar to that in the plane problem, as given by (4) with $\delta>R$. This is plausible because by far the most rapid variation takes place in the topmost layer, where the plane and cylindrical problems are similar. We must bear in mind that the fluid is now decelerated from an initial velocity proportional to the radius. We thus substitute for (4) the equations

$$u(r)=u(a)(r/a)\{(R-r)/(R-a)\}^{1/7}$$
$$=\omega_r r\{(R-r)/R\}^{1/7}, \quad (8.1)$$
$$\tau_0=0.02\rho\{u(a)R/a\}^{7/4}\{s/(R-a)\}^{1/4}$$
$$=0.02\rho\{\omega_r R\}^{7/4}(s/R)^{1/4}. \quad (8.2)$$

Here we have defined the angular speed of the core relative to the mantle, ω_r, as $u(a)/a$ for $a\ll R$. The different aspects of (4.1) and (8.1) are illustrated by the broken lines and the solid lines, respectively, of Fig. 1. The straight line in each case shows the initial state of flow before the deceleration of part of the fluid by the turbulence.

By use of (8.1) in the appropriate integration, we obtain for the angular momentum, per centimeter along the axis, the expression

$$1.2\rho R^4\omega_r.$$

The time rate of change of this angular momentum is equal to the torque per centimeter, $2\pi R^2\tau_0$, as given in (8.2). We thus have the equation

$$\dot{\omega}_r=-10^{-1}s^{1/4}R^{-1/2}\omega_r^{7/4}=-10^{-6}\,\text{sec.}^{-1/4}\,\omega_r^{7/4} \quad (9)$$

with $R=3\times10^8$ cm, $s=10^{-3}$ cm^2 sec.$^{-1}$. By integrating this we obtain

$$\omega_r^{-3/4}=\omega_0^{-3/4}+(3/40)s^{1/4}R^{-1/2}t. \quad (10.1)$$

If we put $t=6\times10^{16}$ seconds, the first term on the right side may be neglected, and we have

$$\omega_r=32s^{-1/3}R^{2/3}t^{-4/3}$$
$$=6\times10^{-15}\,\text{sec.}^{-1}\approx10^{-10}\Omega \quad (10.2)$$

independent of the initial speed of relative rotation, within reasonable limits. The result is also quite insensitive to the value assumed for the coefficient of viscosity. Equation (10.2) corresponds to one revolution of the core within the earth in about 10^{10} days $=3\times10^7$ years.

For the sake of discussing the possible decay of a large difference of angular velocity, we have, above, treated the case in which the mantle has a constant angular velocity, and have found that the difference would in that case be reduced to a small value within the lifetime of the earth. For relative rotations as slow as this result, the angular acceleration of the mantle may no longer be neglected. Still considering only the gradual slowing down of the earth's rotation, and leaving until later the precession, we have for the angular acceleration of the mantle $-5\Omega/T=-6\times10^{-21}$ sec.$^{-2}$. Instead of (9) we would now have

$$-\dot{\omega}=10^{-6}\,\text{sec.}^{-1/4}\,\omega_r^{7/4}, \quad (9')$$

where $\dot{\omega}$ is the angular acceleration of the core in an astronomical coordinate system, and ω_r is again the angular speed of the core relative to the mantle, which is responsible for the torque. If the angular speeds of the core and mantle are to remain very nearly equal, $\dot{\omega}$ must be equal to the angular acceleration of the mantle. The value of ω_r thus determined,

$$\omega_r = 10^{-8}\ \text{sec.}^{-1} = 10^{-4}\Omega \qquad (11)$$

is much greater than (10.2). From these two results we may infer the result of the combined problem. If there were a considerable initial relative rotation in addition to the deceleration of the mantle, the relative rotational speed would approach the value (11) within a time shorter than the life of the earth. The time of approach to this order of magnitude of ω_r may be estimated by putting $\omega_r = 10^{-8}$ sec.$^{-1}$ in (10.1), with the result $t = 3\times 10^4$ years.

The Reynolds number corresponding to (11) is still above 10^{10}, so turbulence would be expected to persist at this speed, as was assumed in the derivation of (11).

Magnetic Drag

The transfer of angular momentum into the core may also be due to the inductive effect of a magnetic field arising within the mantle and penetrating into the metallic core. This effect would be closely analogous to the torque exerted on a suspended copper disk by a rotating magnet, or to the functioning of a magnetic speedometer. There would also be such an inductive drag if the magnetic field should arise in the core and penetrate into a metallic mantle, but this would cause large variations in the observed magnetic field at the surface unless the relative rotation were extremely slow or the field penetrating to the surface should be stabilized by an inductive effect in the mantle. For the sake of simplicity, we shall assume in this section that the magnetic field arises in the mantle and penetrates into the core. An analysis of the geographic variations of the earth's magnetism shows that it may be attributed to a strong central dipole making an angle of $11\frac{1}{2}°$ with the axis of rotation of the earth, and to 14 subsidiary dipoles located at the surface of the core, each only about 1/100 as strong as the central dipole.[4] We shall neglect the irregularities in the field which are described by the subsidiary dipoles and consider only the main part of the earth's field which in the analysis is attributed to the central dipole. A uniformly magnetized spherical shell gives rise to a field outside it identical with the field of a dipole at the center. In our treatment we may, therefore attribute this field to a uniform magnetization of the mantle or of a shell within the mantle, or to an equivalent distribution of current. It might, for example, be caused by a suitable distribution of core-encircling current near the surface of the core. Without an inductive effect, the field in any case would be practically uniform within the core. Its strength would depend upon the location of the magnetization: It would be about as strong as the field at the surface (1 dyne$^{1/2}$ cm^2) if the entire mantle were magnetized, but about 8 times that strong if the magnetization or current were confined to the deepest layers of the mantle. Thus we may take the magnitude of the field in the core as about 10 dyne$^{1/2}$ cm^{-1} or somewhat less. In discussing the gradual deceleration, we may consider that the axis of relative rotation of the core and mantle is the geographic axis. Then only the equatorial-plane component of the field within the core produces an inductive effect, and this component we shall call H_0. Its magnitude is about 2 dyne$^{1/2}$ cm^{-1} or less.

The problem of the rotation of a conducting sphere in a uniform magnetic field is simplest if the sphere is rigid. This simple case is not entirely unrelated to the problem of the earth's core if we consider the simultaneous effects of turbulence and magnetic drag. Turbulence allows most of the slip to take place near the boundary, so that the curved velocity distributions shown in Fig. 1 are somewhat similar to the straight lines associated with rigid motion. Before considering the other extreme case in which only magnetic drag decelerates a fluid core, we first treat the simple case of a rigid core rotating, with free slip at its boundary, in a transverse field H_0. With very slow rotation this field induces in the moving matter of the core a current proportional to the rotational speed. The current interacts with the field in such a way as to produce a decelerating torque which is also proportional to the angular speed. At higher speeds, the induced current effectively opposed the penetration of the field into the core. This

[4] A. G. McNish, Trans. Am. Geophys. Union 2, 287 (1940).

"skin effect" is so strong that the decelerating torque actually decreases with increasing speed at high rotational speeds.

This problem of the rotation of a spherical conductor in a uniform magnetic field has been treated quantitatively by R. Gans, who finds for the decelerating torque[5]

$$P=(3/2)H_0^2R^3\{(2\gamma)^{-\frac{1}{2}}-\gamma^{-1}+\cdots\},\ \gamma \text{ large}, \tag{11.1}$$

$$P=(3/22)H_0^2R^3\gamma\{1-10^{-2}\gamma^2+\cdots\},\ \gamma \text{ small}, \tag{11.2}$$

$$\gamma=4\pi\sigma R^2\omega_r,$$

where σ is the specific conductivity of the core in electromagnetic units, which we assume to be about 10^{-6} sec. cm^{-2} (that is, 10^4 ohm cm, a representative value for metals). The division between "γ large" and "γ small" is about $\gamma=10$, or $\omega_r=10^{-11}$ sec.$^{-1}\approx 10^{-7}\Omega$, a very small fraction of the angular speed of the earth. From the first term of (11.1) and from the equation of motion

$$\dot{\omega}=-(15/8\pi\rho R^5)P, \tag{12}$$

with $\dot{\omega}=\dot{\omega}_r$ because of the steady rotation of the mantle, we obtain

$$\omega_0^{3/2}-\omega_r^{3/2}=4\times 10^{-24}\text{ sec.}^{-5/2}\,t \tag{13.1}$$

with the evaluation of the constants suggested above. In the case of an initial rotation $\omega_0\approx\Omega$ for example, Eq. (13.1) tells us that it would take about 2×10^{17} sec., or about 10^{10} years for $\dot{\omega}$ to become very small. For small values of γ we have instead

$$\ln(\omega_r/\omega_0)=-4\times 10^{-7}\text{ sec.}^{-1}\,t. \tag{13.2}$$

Thus the magnetic drag would stop the relative rotation practically completely within about a year if it should ever become rather slow.

We turn again to the problem of the core within a decelerating mantle. With this magnetic drag, which rises to a maximum and then falls off with increasing ω_r, there are two values of ω_r for which the torque is just sufficient to supply the core with a deceleration equal to that of the mantle. These are obtained by setting the average deceleration of the mantle, 6×10^{-21} sec.$^{-2}$, equal to $-\dot{\omega}$ in (12) for both large and small γ, and are

$$\omega_r=2\times 10^{-7}\text{ sec.}^{-1},\qquad \gamma \text{ large} \tag{14.1}$$

$$\omega_r=1.5\times 10^{-14}\text{ sec.}^{-1},\qquad \gamma \text{ small}. \tag{14.2}$$

For any ω_r between these values, the magnetic drag is great enough to reduce ω_r in spite of the deceleration of the core, and throughout most of that range the drag is great enough that the deceleration is negligible.

It is probable that the violent separation of the earth from the sun left the core and mantle (as soon as they could be distinguished) with an initial relative rotation of the order of magnitude Ω, or somewhat more. Our hypothetical problem in which we ignore the precession is then as close as possible to reality if we consider both an initial relative rotation and the deceleration. We have seen in Eqs. (10.2) and (11) that the torque caused by turbulence is in this case capable of reducing ω_r down to 10^{-8} sec.$^{-1}$. This lies between the limits (14), so the magnetic drag would be strong enough to reduce ω_r further, down to the value given by (14.2), if the rotation were approximately rigid.

In the case of a fluid core accelerated by magnetic drag alone, the penetration of the magnetic field is greatly facilitated by the possibility that the successively deeper layers may each be first penetrated, then accelerated, by the magnetic field. We consider a fluid core initially rotating with angular speed ω_0 in a transverse field H_0 which does not penetrate it initially. The penetration is most rapid where H_0 is tangent to the sphere. There the penetration is governed by the equation, derivable from Maxwell's equations,

$$\dot{H}_x=(4\pi\sigma)^{-1}\partial^2H_x/\partial z^2$$

if z is the axis normal to the boundary. Comparison of this with Eqs. (1) and (3) shows that the depth of penetration of a tangential field into a stationary conductor in time t is

$$z\approx(t/\pi\sigma)^{\frac{1}{2}}. \tag{15}$$

The field at a part of the surface of the rotating core is reversed after a time $t=\pi/\omega_r$, so the depth of penetration when the outer layers rotate with angular speed ω_r is about

$$\zeta=(\omega_r\sigma)^{-\frac{1}{2}}.$$

The lines representing the field then slope

[5] R. Gans, Archiv f. Elektrotechnik 9, 413 (1921).

inward from the surface to this depth as they wind $\frac{1}{4}$ of the way around the sphere, so the average radial component of the field is about

$$H_z = H_0\zeta/R.$$

The angular acceleration of the shell of thickness ζ caused by the currents induced by H_z is then approximately

$$\dot{\omega}_r = (\sigma\zeta^2 H_0^2/\rho R^2)\omega_r = H_0^2/\rho R^2 = \Omega/(2\times 10^{13}\text{ sec.}).$$

We may then say that the time required to decelerate a layer of thickness ζ from $\omega_0 = \Omega$ to rest is about 2×10^{13} sec. With $\omega_r \approx \Omega$, we have $\zeta \approx 10^5$ cm. If we should assume that the deceleration of the next deeper layer would not begin until this outer layer is completely decelerated, we might then estimate roughly that it would require a time

$$(R/\zeta)2\times 10^{13}\text{ sec.} = 6\times 10^{16}\text{ sec.} = T$$

for the deceleration to penetrate to the center. This estimate is somewhat too large, both because a smaller value of the radius should be used for the inner layers and because the deceleration of one layer will commence before the layer above it is brought to rest.

With only magnetic drag, the penetration would then be complete within the lifetime of the earth, T. The angular speed of the core relative to the mantle which is necessary to transmit to any layer of the core the average deceleration, $\dot{\omega} \approx 5\Omega/T$, of the mantle, is then

$$\omega_r \approx (\rho/\sigma H_0^2)\dot{\omega} \approx (5\rho/\sigma H_0^2 T)\Omega \approx 2\times 10^{-10}\Omega$$

in agreement with (14.2) for the rigid case. Since we obtain the same result for ω_r in both extremes, it may be expected to apply in order of magnitude to the intermediate case in which we have turbulent drag in addition to magnetic drag. This ω_r, or even that in Eq. (11) given by turbulence alone, is so much less than Ω that we may take the angular speeds of the mantle and core to be equal, and turn to consideration of the precession.

The 27,000-Year Precession

We have as yet been considering only the smaller component of the vectorial angular acceleration of the earth. The magnitude of the angular acceleration associated with the 27,000-year precession about a 24° cone is 10^{-1} year^{-1} $\Omega = 2\times 10^{-16}$ sec.$^{-2}$. This is about 4×10^4 times as great as the deceleration discussed above. The angular acceleration of the mantle is thus represented by a vector $\dot{\boldsymbol{\omega}}_m$ about normal to its angular velocity $\boldsymbol{\omega}_m$. We shall now investigate whether the turbulent and magnetic interaction between the core and mantle might be strong enough to cope with this larger component of the angular acceleration also.

If $\boldsymbol{\omega}$ and $\boldsymbol{\omega}_m$ should have both different magnitudes and considerably different directions, as in Fig. 2(a), we may assume that $\boldsymbol{\omega}_r$ may be divided, at least roughly, into two parts: a part, either normal to $\boldsymbol{\omega}$ or making equal angles with $\boldsymbol{\omega}$ and $\boldsymbol{\omega}_m$, which causes a turbulent interaction between the core and mantle tending to reduce the angle α between $\boldsymbol{\omega}$ and $\boldsymbol{\omega}_m$; and a part parallel to $\boldsymbol{\omega}$, which causes an interaction tending to reduce the magnitude of $\boldsymbol{\omega}$, as discussed in the preceding section. Although this assumption is intuitively rather plausible as an approximation, it is not obvious, both because of the nonlinear relation between relative velocity and turbulent drag, and because of the necessity of discussing relative velocity in a rotating coordinate system. The nonlinearity would be expected to decrease the decay times of the relative rotation estimated below. Before we make such estimates, we shall discuss the relative velocity in more detail in the following section.

The usual discussion of fluid friction, laminar or turbulent, depends on a persistent motion of the mass of the fluid in one direction, relative to a boundary which is at rest in an inertial coordinate system. In our case, the relative motion which is responsible for the interaction is the difference of the motions of two bodies neither of which is even approximately unaccelerated relative to an inertial system. The following discussion of some details of the motion will form a basis for a plausible extension of the usual treatment.

Relative Motion in the Precession

We have seen that the interactions are of an order of magnitude to cause a change of angular velocity comparable to $\boldsymbol{\Omega}$ only in many years: $\boldsymbol{\omega}$ and $\boldsymbol{\omega}_m$ are practically constant for times as short as a few days. For such short times, the relative velocity $\boldsymbol{\omega}_r$, which is the difference of $\boldsymbol{\omega}$

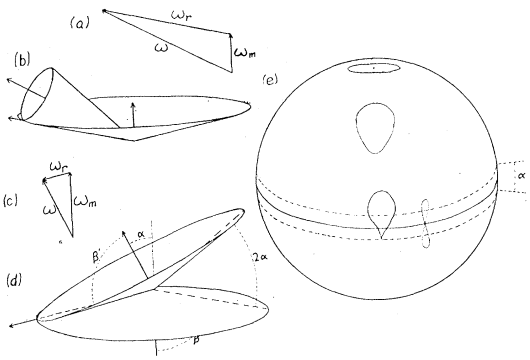

FIG. 2. (a) and (b) describe a possible initial rotation of the core and mantle with quite different angular speeds. (c) and (d) describe the probable present rotation with the angle between the axes exaggerated. On the sphere (e) are shown the paths, fixed relative to the mantle, followed by representative points of the core.

and $\boldsymbol{\omega}_m$ and is parallel to neither, is practically constant in space and therefore rotates with a period of one day in the coordinate system of the mantle. As the mantle rotates about the axis of $\boldsymbol{\omega}_m$, the direction of $\boldsymbol{\omega}_r$ sweeps out a cone in the mantle-fixed coordinate system. The axis of the cone is the polar axis. Likewise $\boldsymbol{\omega}_r$ sweeps out a cone whose axis is the axis of rotation of the core in the core-fixed coordinate system. Since these two cones always have a line of contact, which is the instantaneous axis of rotation, the motion of the core relative to the mantle may be described as the rolling of the core-fixed cone on the mantle-fixed cone.

The shapes of the cones, of course, vary with the angles between the vectors $\boldsymbol{\omega}$, $\boldsymbol{\omega}_m$ and $\boldsymbol{\omega}_r$. In the extreme case in which $\boldsymbol{\omega}$ is considerably greater in magnitude than $\boldsymbol{\omega}_m$, with a considerable angle between them, the angle between $\boldsymbol{\omega}$ and $\boldsymbol{\omega}_r$ is comparatively small (Fig. 2(a)), and the core-fixed cone is a rather narrow one, although the mantle-fixed cone may be rather wide, as shown in Fig. 2(b). In the limit in which the core-fixed cone becomes very narrow, the motion of the core relative to the mantle is a fast rotation with a very slow change in the general direction of the axis, relative to the mantle. In this limit the motion is not essentially different from that discussed in the preceding section, and the result of that discussion leads us to expect that $\boldsymbol{\omega}$ would decrease until it is not much larger than $\boldsymbol{\omega}_m$ in a time much shorter than the age of the earth.

In the case in which $\boldsymbol{\omega}$ and $\boldsymbol{\omega}_m$ are equal in magnitude, and when they have an angle α between them considerably less than $\pi/2$, the cones are equal and very wide, with apex angle $\pi-\alpha/2$, as in Fig. 2(c) and (d). In describing the motion of the core relative to the mantle, we may think of the lower cone in Fig. 2(d), the mantle-fixed cone, as stationary, with the upper cone rolling on it. A point on the core-fixed cone describes a closed glass-drop shaped figure with a cusp at the point of contact with the mantle-fixed cone, as sketched on the sphere in Fig. 2(e). The points on the polar axis of the core describe circles. A point above the cone but not on the axis describes a figure intermediate between the glass-drop shape and a circle, a pear-shaped figure without a cusp. A point between the core-fixed cone and the equatorial plane of the core describes a figure-of-eight. The diametrically opposite points trace the same figures. The motion of points on the inner surface of the mantle relative to the core is of course similar; they trace closed paths, of over-all north-and-south dimensions $2\alpha R$. The velocity of the surface of the mantle past a given point near the surface of the core rotates periodically into all directions of the compass, in the core-fixed coordinates.

If the core-fixed coordinate system were an inertial system, this would mean that the impulse imparted to the upper layers of the core during one half-cycle would be largely canceled by the impulse imparted during the next half-cycle. There would thus be comparatively little penetration of the turbulent flow, for example, and no net drag on the interior of the core. This cancellation is not very effective in the rotating core because of the Coriolis forces in the rotating coordinate system. In order to see that the Coriolis forces, which are a way of expressing the conservation of momentum and of angular momentum, act in the right way, let us consider a model in which an intermediate layer between the core and mantle originally rotates with the core, and has a frictional interaction with the mantle which may be turned on and off at will. We shall consider as typical the interaction, with the mantle, of the point which traces the pear-shaped figure in Fig. 2(e). During the instant represented by Fig. 2(d) and (e), this point is moving up the left side of the pear, so the adjacent surface of the mantle moves down past it. Let the interaction be turned on instantaneously at this time. This impulse makes an increment of angular velocity of the intermediate layer, $\mathbf{d\omega}$, in the direction opposite to $\boldsymbol{\omega}_r$. We next turn the interaction on instantaneously a half-cycle later, when the point is moving down the right side of the pear, and the mantle up past it. In a space-fixed coordinate system, both of the cones in Fig. 2(d) are spinning in such a way that their line of contact remains stationary, with no slipping, and the sphere in Fig. 2(e) rotates with the core-fixed cone. At this second instant, the pear is around behind the sphere. In the space-fixed system, the conservation of angular momentum maintains the first increment of angular velocity $\mathbf{d\omega}$ intact, and the second increment is in the same direction, so the two are additive. In the core-fixed coordinates it is the Coriolis force which maintains $\mathbf{d\omega}$. Thus we see that the forces acting during the different parts of the periodic relative motions do not effectively cancel one another.

A third case to be discussed is similar to the last, but with $\boldsymbol{\omega}$ slightly greater than $\boldsymbol{\omega}_m$, so that $\beta > \beta'$ in Fig. 2(d). In this case the circumference of the mantle-fixed cone (at a radius R) is greater than the circumference of the core-fixed cone and points in the mantle-fixed cone advance relative to the core by an angle $2\pi(\sin\beta - \sin\beta')$ during each revolution of the core, instead of returning to the same point and making a closed figure. Other points on the inner surface of the mantle similarly fail to make closed figures, and progress circuitously and slowly around the core. Their paths are somewhat similar to trochoids (generalized cycloids). Just as in the case of an electron following a trochoidal path in crossed magnetic and electric fields, we can speak of a net velocity of translation after averaging out the irregularities of the path, so here we can speak of a net angular speed with which the points of the core-fixed cone, for example, effectively rotate in the mantle-fixed system. This net relative angular speed is the scalar quantity

$$\omega(\sin\beta - \sin\beta')/\sin\beta = (\omega - \omega_m). \qquad (16)$$

The first member is derived by considering the gradual progress of the cusp (Fig. 2(e)) about the lower cone in Fig. 2(d), and the second member is obtained from the first by use of the law of sines in Fig. 2(c). Thus points on most of the surface of the mantle circulate around the core with a net relative angular velocity whose magnitude is approximately the difference of the angular speeds and whose direction is that of $-\boldsymbol{\omega}$. This may be expected to be effective in decelerating the core, in keeping with the assumption that the two components of $\boldsymbol{\omega}_r$ may be considered separately.

Turbulent Force in the Precession

We next investigate whether the forces are adequate in order of magnitude to reduce α to a small value. Although the two processes would probably take place at the same time from general initial conditions, we may consider that the core has already been decelerated so that $\omega = \omega_m$ in magnitude. We then have the second case discussed above, with $\beta = \beta'$ in Fig. 2(c) and (d). Having seen that the vacillating direction of the relative motion in the core-fixed coordinate system does not bring about an effective cancellation of the frictional impulses imparted at various times, we assume that we may obtain a sufficient approximation to the torque effective in the astronomical coordinate system from the relative motion by use of Eq. (8.2) above. In making this assumption, we take into account, in the core-fixed system, the Coriolis forces due to the horizontal average motion of the various layers of fluid near the surface. We have no way of showing that the vortex-like fluctuations from this average, which describe the turbulence itself, would not behave quite differently, under the influence of the Coriolis forces and the closed-path motion of Fig. 2(e), than in the simple case. Qualitatively, one might expect large-scale turbulence to be less fully developed here than in a corresponding rotation about a steady axis. Such an effect would tend to make the lag between the axes larger than estimated below, but, even if extreme, would not make it larger than 24°. The inaccuracies discussed in footnote 3 tend in the opposite direction.

Even a comparatively small interaction, one capable of changing $\boldsymbol{\omega}$ by as much as Ω in less than the lifetime of the earth, would gradually bring $\boldsymbol{\omega}$ into coincidence with the axis of the 24° cone about which $\boldsymbol{\omega}_m$ precesses. But it requires an interaction which may have a considerable effect in less than 27,000 years to bring $\boldsymbol{\omega}$ closer to $\boldsymbol{\omega}_m$ and thus make α remain less than 24°. Starting with $\boldsymbol{\omega}$ along the axis of the 24° cone, the time during which the rate of change of $\boldsymbol{\omega}$ remains in the same general direction is about $27{,}000/2\pi$ years $=1.3\times 10^{11}$ sec. This value of t in Eq. (10.1) gives $\omega_r=2\times 10^{-7}$ sec.$^{-1}=3\times 10^{-3}\Omega$. This shows that the interaction is quite strong enough to make $\boldsymbol{\omega}$ approximately parallel to an element of the cone, rather than stay near the axis of the cone. How closely it may keep up with $\boldsymbol{\omega}_m$ in its precession may be found by substituting $|\dot{\boldsymbol{\omega}}|$ for $-\dot{\omega}$ in Eq. (9′) and putting it equal to the magnitude of $\dot{\boldsymbol{\omega}}_m$ in its precession, 2×10^{-16} sec.$^{-2}$. One finds

$$\omega_r=3\times 10^{-6}\text{ sec.}^{-1}=\Omega/25.$$

Under the influence of turbulence alone, the axis of the core is thus expected to lag behind the axis of the mantle in the precession by an angle of about 2°. The relative angular velocity is great enough that it would correspond to about one revolution of the core inside the mantle in 25 days, if it were not for its rapid change of direction relative to the mantle. Because of its rapid change of direction, the motion of the core inside the mantle, as here estimated, can be described as the rolling of two cones on one another, as in Fig. 2(d) with $\alpha\approx 2°$. Each point in the core-fixed coordinate system then describes in the mantle-fixed coordinates a small closed figure daily (Fig. 2(e)). Near the surface of the core these figures have linear dimensions of about the order of magnitude $10^{-1}R=300$ km.

Magnetic Drag in the Precession

The rapid change in the direction of the axis of rotation of the core in the mantle-fixed coordinate system completely alters the treatment of the magnetic drag. The earlier treatment considered the rotation of a conducting sphere about a fixed axis in a homogeneous field. In that case, only the component of the field normal to the axis could cause a drag, and the penetration of this component into the sphere was opposed by the induced currents, because of the steady flow of conducting matter across the lines along which the field seeks to establish itself. In the motion of the core relative to the mantle described in the preceding paragraph, the motion of the conducting matter is not steady, but is a periodic motion about small closed figures. Such a motion is not effective in opposing the penetration of the field. The time of penetration would be about the same as if the core were stationary. It may be estimated by putting $z=R$ in Eq. (15). One obtains, for the time of penetration to the center of the earth, about 3×10^{11} sec. $=10^4$ years, which is much less than the age of the earth. This agrees in order of magnitude with the fact noted above that the division between "γ large" and "γ small," which is presumably the division between incomplete and complete penetration, lies at $\omega_r=10^{-11}$ sec.$^{-1}$. We thus expect the magnetic field of the mantle to penetrate through the core from north to south.

We shall make a rough estimate of the effect of this penetrating field on the motion of the core. If this is at all comparable with the effect of the torque caused by turbulence, so that we are left with a motion of points of the core about closed paths with a period of one day, we should expect the lines representing the field to bend periodically near the surface of the core in such a way as to allow the lines deep within the core to follow the motion of the core rather closely. The angle at which the lines cross the surface of the core near the pole may be estimated from the requirement that the field supply an angular acceleration $|\dot{\omega}|=2\times 10^{-16}$ sec.$^{-2}$, as demanded by the precession. The tangential force per unit area applied to a surface by a field H penetrating the surface at an angle θ to the normal is found from the Maxwell magnetic stress tensor to be $(H^2/4\pi)\sin\theta\cos\theta$. Since the integrated normal component is not changed by the bending of the field at the surface, we may on the average say that the normal component of the earth's field at the surface of the core is $H_n=H\cos\theta$, and we have placed an upper limit to its magnitude at the poles, 10 dyne$^{1/2}$ cm^{-1}. The average value of H_n^2 over the surface is one-third of the value at the poles. For the magnetic torque on the core we obtain

$$\tfrac{1}{3}H_n^2R^3\sin\theta/\cos\theta,$$

where H_n is the value at the poles. For this to be equal to $I|\dot{\omega}|$ in the precession, θ must satisfy

$$\tan\theta \approx (8\pi/5)\rho|\dot{\omega}|R^2/H_n^2 \approx 10 \qquad (17)$$

with the evaluation of constants suggested above.

Before applying this to the fluid core, let us consider a hypothetical case in which a rigid core is free to rotate with easy slipping at the surface. If this is accelerated by the magnetic drag, to keep up with the precession, the depth of penetration of the horizontal component of H is estimated, by putting $t=\frac{1}{2}$ day $=4\times10^4$ sec. in (15), to be about 10^5 cm. The angle α by which the core must lag behind the mantle in order that the magnetic drag shall supply the required acceleration is then roughly equal to this depth multiplied by $R^{-1}\tan\theta$, or $\alpha\approx3\times10^{-3}$. (In this connection, one may note that the error function $\Theta(q)$ in Eq. (2) is almost linear in q nearly up to the value $q=1$ which we have taken to correspond to the depth of penetration.) Such a small value of α is, of course, largely caused by the rigidity which was arbitrarily introduced in this case. But it is instructive to observe that the transverse motion of the magnetic lines through the matter is very small, even in this extreme case in which the induction in the outer skin of the core, less than $10^{-3}R$ thick, is responsible for the angular acceleration of the entire core.

In the case with only magnetic drag accelerating a fluid core, which we consider next, we may, therefore, neglect the transverse motion of the magnetic lines through the matter. In this case the axes of rotation of the successively deeper spherical shells of the core are expected to lag behind the axis of the mantle by successively greater angles, but, when an equilibrium has been reached in which these angles are constant in time, the torque required to maintain a steady angular acceleration of the core is the same as in the rigid case. Thus, even though the core is not a rigid body, (17) applies to this case also. Moreover, the analogous equation

$$\tan\theta(r) \approx (8\pi/5)\rho|\dot{\omega}|r^2/H_n^2 \approx 10r^2/R^2 \qquad (17')$$

serves to determine the inclination $\theta(r)$ of the magnetic lines at any radius r within the core. The angle of lag in the vicinity of the center is

$$\alpha = \int_0^R r^{-1}\tan\theta(r)dr \approx 5. \qquad (18)$$

The lines would spiral inward with almost one complete revolution if $\boldsymbol{\omega}_m$ should precess in a plane. The magnetic drag alone would thus leave the interior rotating about the axis of the 24° cone. This value of α, which is necessary to make the magnetic drag sufficiently strong, is so much larger than the value 1/25 estimated in the case of turbulence alone that the mere *superposition* of the magnetic drag cannot be expected to modify that result appreciably.

Here we have not considered the possible *interaction* of the magnetic force and the turbulence—the manner in which the vortex motion within the turbulent flow might be modified by the presence of the magnetic field. In treating the possible penetration of the field through the vortex motion, we may consider that the angular speed of the eddies is at least of the order of magnitude of ω_r, say 10^{-6} sec.$^{-1}$. Then the parameter γ for the eddy, as discussed for the entire cone in connection with (11) above, is large if the size of the eddies is as great as 10^6 cm $=3\times10^{-3}R$. Since the eddies are probably larger than this, it seems unlikely that the field penetrates the eddies enough to make the interaction appreciable.

In the present state of geophysics, it seems impossible to be sure that one's hypotheses concerning the deep interior of the earth correspond to reality. An attempt has been made to discuss the situation which at present seems most plausible, but it may be that the results will merely serve in some way to prove—perhaps to disprove—the hypotheses. Our main result is that, if the core is a fluid at all similar to molten metals, the bulk of it has an angular velocity relative to the mantle, in order of magnitude a few percent as fast as the earth's rotation itself, in spite of the force due to turbulence and the smaller forces of direct viscosity and of magnetic induction.

The very helpful suggestions of Professor E. Teller and Dr. Ross Gunn have been appreciated. This investigation is, in fact, a sequel to a series of conversations with them on possible sources of the earth's magnetism, and to discussions at the Sixth Washington Conference on Theoretical Physics.