The Density Distribution in the Steady-State Diffusion of Neutrons

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By carrying the method of Laplace transformations and inversions to its ultimate conclusion, an expression in closed form for the spatial distribution in density of diffusing particles is obtained. This derivation is valid both for the case of incident currents of particles upon the face of a half-infinite medium and for the case of source distributions within that medium. The final expressions are given in terms of a transcendental function and its zeros together with a definite integral, all of which require numerical evaluation. Such numerical calculations have not been performed, since the problem is principally of theoretical interest in the sense that these methods yield for the first time an exact nonseries solution of a well-known, and in this respect, unsolved integral equation. From an experimental point of view it is the ratio of activities at different points within the medium rather than the ratio of particle densities which is of principal interest. These quantities are not the same, and some results for the former based on the exact theory have already been given.

INTRODUCTION

THE problem of neutron diffusion under steady-state conditions has been solved for the emergent distribution from a medium, and the results have been published in previous communications.[1,2] The study of this problem has now been carried one step further in order to find exact expressions for the particle density within the medium in which the diffusion occurs.

The idealizations employed in the statement of the problem are discussed fully in the previous papers.[1,2] We may mention briefly that the diffusion occurs under steady-state conditions within a half-infinite, isotropic, homogeneous medium having a plane boundary face, that the diffusing particles interact isotropically and without the exchange of energy with particles of the medium, and that there are no mutual interactions. Furthermore, the incident current is taken to have azimuthal symmetry about the perpendicular to the boundary face, and to vary only in polar angle, whereas the source distribution is taken isotropic and to vary only in the direction perpendicular to the boundary plane. Under these circumstances the particle velocity enters only parametrically, and the number of independent variables is reduced to two, (x, z), where x is a spatial coordinate measured into the medium and positively to the right from the boundary plane as origin, and $z=\cos\theta$ is the direction cosine of the particle velocity with respect to the positive x axis. The distribution function $f(x, z)$ is taken to represent the density of particles at x per unit solid angle in velocity with the velocity vector lying at z. The emergent distribution of particles from the medium is then represented by $f(0, z)$ with $-1\leq z<0$.

THE EMERGENT DISTRIBUTION

It has been shown[1] that the emergent distribution is given by the value on the real interval $z=(-1, 0)$ of the function

$$f(0, z)=\frac{n\zeta}{\zeta-z}\frac{1}{p_1(z)p_2(\zeta)}\quad \begin{matrix} R(z)<0 \\ R(\zeta)\geq 0\end{matrix} \tag{1}$$

which is analytic over the entire negative half of the z plane. This result is valid for a particular form of incident current and source distribution, but the validity of superpositions renders the solution for other forms readily available. For a source distribution of the form

$$q(x)=qe^{-x/l\zeta}\quad R(\zeta)\geq 0$$

the constant n is given by

$$n=ql/v, \tag{1a}$$

where l is the total m.f.p. of diffusing particles. For an incident current of the form

$$\begin{matrix} f_0(z)=f\delta(z-\zeta) & 0<\zeta\leq 1, \\ n=f\sigma & \sigma=l/2l_s, \end{matrix} \tag{1b}$$

[1] E. A. Schuchard and E. A. Uehling, Phys. Rev. **58**, 611–623 (1940). Further references to this paper will be to SU.
[2] O. Halpern, R. Lueneburg and O. Clark, Phys. Rev. **53**, 173–83 (1938).

when l_s is the m.f.p. for scattering, which together with the capture m.f.p. l_c contributes to l according to the relation $1/l=1/l_s+1/l_c$. This equivalence of source and incident current distributions insofar as the emergent current is concerned and for values of ζ in the interval (0, 1) has been discussed in SU. The earlier papers[1,2] may be consulted also for a discussion of the functions p_1 and p_2. It will be sufficient for our purpose here to mention only that

$$\left.\begin{aligned} \ln p_1(z) &= -\frac{z}{\pi}\int_0^\infty dt\frac{\ln p(it)}{z^2+t^2} \quad R(z)<0 \\ \ln p_2(z) &= \frac{z}{\pi}\int_0^\infty dt\frac{\ln p(it)}{z^2+t^2} \quad R(z)>0 \end{aligned}\right\}, \tag{2}$$

$$\left.\begin{aligned} &p_1(z)=p_2(-z) \\ &p(z)=p_1(z)p_2(z)=1+\sigma z\int_{-1}^{1}\frac{d\alpha}{\alpha-z} \end{aligned}\right\}, \tag{3}$$

and that $p_1(z)$ has the branch point and zero of $p(z)$ in the positive half-plane, while $p_2(z)$ has the branch point and zero of $p(z)$ in the negative half-plane. These branch points lie at ± 1, and the zeros lie at $\pm z_0$ where

$$p(z_0)=1+\sigma z_0\int_{-1}^{1}\frac{d\alpha}{\alpha-z_0}=0.$$

One finds that $1\leqq z_0\leqq\infty$. The value of z_0 is determined only by σ, being unity for $\sigma=0$, and becoming infinite for σ approaching its maximum value $\frac{1}{2}$.

An obvious property of $p(z)$ is

$$\lim_{|z|\to\infty} p(z)=1-2\sigma.$$

This result shows that in the integrated form

$$p(z)=1+\sigma z\ln\left[(z-1)/(z+1)\right] \tag{4}$$

with z restricted to points off the real interval $(-1, 1)$, the principal value of the logarithm must be taken. If, furthermore, we write

$$p(z)=1+\sigma z(\ln\rho+i\alpha), \tag{5}$$

where

$$\rho=\left|\frac{z-1}{z+1}\right|=\left|\left(\frac{(u-1)^2+v^2}{(u+1)^2+v^2}\right)^{\frac{1}{2}}\right|$$

$$\tan\alpha=2v/(u^2+v^2-1)\quad z=u+iv$$

the same condition requires the vanishing of α at infinity. Thus, one has for α in the neighborhood of the branch points the values consistent with the arrangement given in Fig. 1 instead of the

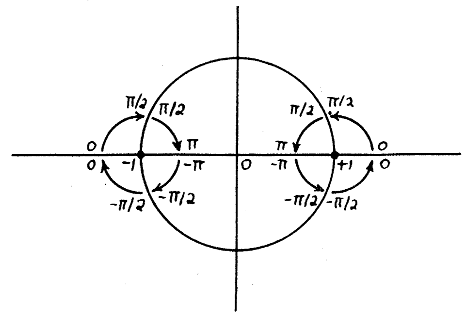

FIG. 1.

alternative arrangement in which 0 and π are substituted for $-\pi$ and 0, respectively, below the real axis.

THE PARTICLE DENSITY

The total density of particles at the plane x is given by

$$F(x)=2\pi\int_{-1}^{1}f(x,z)dz.$$

It has been shown in SU that the density function must satisfy a well-known inhomogeneous integral equation

$$F(x)=g(x)+\frac{1}{2l_s}\int_0^\infty F(x')K\left(\frac{|x-x'|}{l}\right)dx', \tag{6}$$

where

$$\begin{aligned} g(x)=2\pi&\int_0^1 f_0(z)e^{-x/lz}dz \\ &+\frac{2\pi}{v}\int_0^\infty q(x')K\left(\frac{|x-x'|}{l}\right)dx', \end{aligned} \tag{7}$$

$$K(s)=\int_0^1 e^{-s/z}\frac{dz}{z}=\int_1^\infty e^{-st}\frac{dt}{t}=-li(e^{-s}),\quad s>0$$

and $f_0(z)$ and $q(x)$ are the incident current and isotropic source distributions, respectively. An integral equation of this type with the symmetric logarithmic integral function as kernel has been encountered in the study of numerous other problems in physics.[3] Particularly, in connection

[3] V. A. Kostitzen, Mem. Sci. Math. **69** (1935). A discussion of the Kundt and Warburg problem of velocity distribution between two parallel plates in relative motion, the Chwolson and Lommel problems of semitransparent sheets and others is given, together with an extensive bibliography. References to some of these papers will be found also in SU.

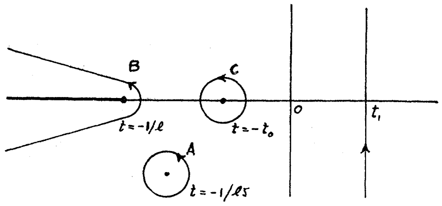

FIG. 2.

with the astrophysical problem of radiant energy transmission through stellar atmospheres, this equation has been the object of considerable study. But no exact solution in closed form has ever been found. Such a solution is required for the case of neutron diffusion, since the development parameter, $2\sigma = l/l_s$, of the iterative solutions is very nearly equal to unity, and such solutions converge very slowly.

It has been shown in SU, Eq. (4a), that $-zf(0, z)$ is the Laplace transform in the variable $-1/lz$ of

$$F(x)/4\pi l_s + q(x)/v$$

for $R(z) < 0$. Introducing $z = -1/lt$ this relation is put in the form

$$\frac{f(0, -1/lt)}{lt} = \int_0^\infty e^{-xt}\left[\frac{F(x)}{4\pi l_s} + \frac{q(x)}{v}\right] dx \quad R(t) > 0.$$

Since $F(x)$ and $q(x)$ may be assumed continuous and the integral is absolutely convergent, this transformation has the unique inversion[4]

$$F(x) = -\frac{4\pi l_s q(x)}{v} + \varphi(x), \tag{8}$$

$$\varphi(x) = \frac{2l_s}{il}\int_{t_1-i\infty}^{t_1+i\infty} e^{xt} f(0, -1/lt)\frac{dt}{t}$$

$$t = t_1 + it_2 \quad t_1 > 0, \tag{9}$$

where t_1 is a line parallel to the imaginary axis and lying within a strip of the t plane within which the integrand is analytic. Reference to Eqs. (1) and (2) shows that this region is the entire positive half of the t plane not including the imaginary axis. Considering now the problems of incident currents and source distributions separately, one has in the first case

$$f_0(z) = f\delta(z-\zeta) \quad q(x) = 0 \quad 0 < \zeta \leq 1,$$
$$g(x) = 2\pi f e^{-x/l\zeta}; \quad F(x) = \varphi(x), \tag{10}$$

where $g(x)$ is the inhomogeneous term of Eq. (6), $\varphi(x)$ is given by Eq. (9), and $f(0, -1/lt)$ is given by Eq. (1) with $n = f\sigma$. In the second case one has

$$q(x) = qe^{-x/l\zeta}, \quad f_0(z) = 0, \quad R(\zeta) \geq 0,$$
$$g(x) = \frac{2\pi ql\zeta}{v} e^{-x/l\zeta}\ln\frac{\zeta-1}{\zeta+1} - \frac{2\pi ql\zeta}{v}\int_0^1 \frac{e^{-x/lz}}{\zeta - z}dz,$$
$$F(x) = -(4\pi ql_s/v)e^{-x/l\zeta} + \varphi(x), \tag{11}$$

where, again, $\varphi(x)$ is given by Eq. (9) and $f(0, -1/lt)$ by Eq. (1) but with $n = ql/v$. The equivalence of incident currents and distributed sources, which holds for the emergent distribution when $0 < \zeta \leq 1$ and $ql/v = f\sigma$, is thus modified slightly for the density function. One finds $F(x)$ for sources to be less than $F(x)$ for incident currents by an amount equal to

$$(4\pi ql_s/v)e^{-x/l\zeta} = 2\pi f e^{-x/l\zeta}.$$

This quantity is, however, just the contribution to the density at the plane x due to the incident current when account is taken of the kinetic theory probability that a particle incident on the boundary will travel the distance x/ζ without collision. This result may be established, also, directly from the integral equation (6) itself.

Thus, for the determination of $F(x)$ for the case either of incident currents or source distributions, one requires the evaluation of the single function $\varphi(x)$ for all values of ζ such that $R(\zeta) \geq 0$. Introducing Eq. (1) into (9), using the relation $p_1(z) = p_2(-z)$, and introducing $\sigma = l/2l_s$ one has

$$\varphi(x) = \frac{nl\zeta}{i\sigma p_2(\zeta)}\int_{t_1-i\infty}^{t_1+i\infty}\frac{e^{xt}dt}{(1+lt\zeta)p_2(1/lt)}. \tag{12}$$

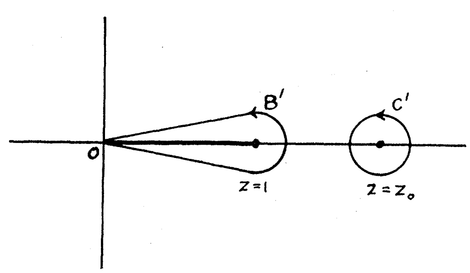

FIG. 3.

[4] G. Doetsch, *Theorie und Anwendung der Laplace Transformation* (Springer, 1937), pp. 40–3.

The integrand, while analytic everywhere to the right of the imaginary axis, has certain singularities in the negative half-plane, which will now be discussed. There exists, first of all, a simple pole at $t=-1/l\zeta$. There is, in the second place, a branch point at $t=-1/l$. The branch cut for $p_2(1/lt)$ may be drawn from the point $t=-1/l$ to the left along the negative real axis to $-\infty$. Finally, there is the zero of $p_2(1/lt)$. This zero is located on the negative real axis at a point $t=-t_0$ where $0\leq t_0\leq 1/l$. Since this zero of p_2 is the zero of p in the negative half of the plane, it may be found, by using Eq. (4) and the subsequent discussion, from the relation

$$p(z_0)=1+\sigma z_0\ln\left[(z_0-1)/(z_0+1)\right]=0,$$

$$1=\sigma z_0\ln\frac{z_0+1}{z_0-1}=2\sigma\left[1+\frac{1}{3z_0^2}+\frac{1}{5z_0^4}+\cdots\right], \tag{13}$$

$$t_0=1/lz_0.$$

$\varphi(x)$ FOR COMPLEX VALUES OF ζ

The function $\varphi(x)$ may be evaluated by deforming the path of integration about the singularities and branch cut in the negative t plane. If ζ is complex, including pure imaginary, or if real it is at least greater than unity and not equal to $1/lt_0$, $\varphi(x)$ may be expressed as the sum of three contour integrals

$$\varphi(x)=(nl\zeta/\sigma)(A+B+C), \tag{14}$$

where A, B and C are the integrals

$$\frac{1}{ip_2(\zeta)}\int\frac{e^{xt}dt}{(1+lt\zeta)p_2(1/lt)}$$

taken about the contours A, B, and C, respectively, of Fig. 2.

Since the contour A contains only a simple pole at $t=-1/l\zeta$, the value of A is obtained immediately. One finds

$$A=\frac{2\pi i}{l\zeta}\frac{e^{-x/l\zeta}}{ip_2(\zeta)p_2(-\zeta)}=\frac{2\pi e^{-x/l\zeta}}{l\zeta p(\zeta)} \tag{15}$$

after introducing the relations given by Eq. (3).

Before evaluating B and C it is convenient to transform back to the z plane by introducing $t=-1/lz$. Introducing Eq. (3) again, one finds that B and C are given by the integrals

$$\frac{1}{ilp_2(\zeta)}\int\frac{e^{-x/lz}}{z-\zeta}\frac{p_2(z)}{p(z)}\frac{dz}{z}$$

over the contours B' and C', respectively, of Fig. 3. The branch cut along the interval $(-1/l,-\infty)$ of the t plane is transformed to the interval (0, 1) of the z plane. Since the function $p_2(z)$ is analytic everywhere for $R(z)>0$, and the contours B' and C' do not inclose the point $z=\zeta$, these integrals are taken only about singularities of $p(z)$, its branch cut and zero, respectively.

The evaluation of B may now be carried out. Permitting the contour B' to collapse upon the branch cut, and referring to Eq. (5) and Fig. 1 for the selection of α in $p(z)$ one obtains for this integral

$$\begin{aligned}B&=\frac{1}{ilp_2(\zeta)}\int_0^1\frac{e^{-x/lu}}{u-\zeta}\frac{p_2(u)}{1+\sigma u(\ln\rho-i\pi)}\frac{du}{u}\\&\quad+\frac{1}{ilp_2(\zeta)}\int_1^0\frac{e^{-x/lu}}{u-\zeta}\frac{p_2(u)}{1+\sigma u(\ln\rho+i\pi)}\frac{du}{u}\\&=\frac{2\pi\sigma}{lp_2(\zeta)}\int_0^1 e^{-x/lu}\frac{p_2(u)}{(1+\sigma u\ln\rho)^2+(\pi\sigma u)^2}\frac{du}{u-\zeta},\end{aligned} \tag{16}$$

where

$$\ln\rho=(1-u)/(1+u).$$

Finally, we may determine C. Since the only singularity within the contour C' is the zero of $p(z)$, the behavior of this function in the neighborhood of $z=z_0$ must be found. This zero lies on the real axis, and we may set $z_0=u_0\geq 1$ where

$$p(u_0)=1+\sigma u_0\ln\left[(u_0-1)/(u_0+1)\right]=0.$$

Using this condition on u_0 one obtains

$$p(u_0+\Delta z)=\Delta z\frac{1-u_0^2(1-2\sigma)}{u_0(u_0^2-1)}+\cdots.$$

Therefore, $1/p(z)$ has a simple pole at $z=u_0$ and its residue is

$$u_0(u_0^2-1)/\left[1-u_0^2(1-2\sigma)\right].$$

Then

$$C=\frac{2\pi}{lp_2(z)}\frac{(u_0^2-1)}{1-u_0^2(1-2\sigma)}\frac{p_2(u_0)}{u_0-\zeta}e^{-x/lu_0}. \tag{17}$$

Bringing these results together by introducing Eqs. (15), (16) and (17) into Eq. (14) one obtains

$$\varphi(x)=\frac{2\pi n}{\sigma}\bigg[\frac{e^{-x/l\zeta}}{p(\zeta)}$$
$$-\frac{\zeta}{\zeta-u_0}\frac{(u_0^2-1)}{1-u_0^2(1-2\sigma)}\frac{p_2(u_0)}{p_2(\zeta)}e^{-x/lu_0}$$
$$-\frac{\sigma\zeta}{p_2(\zeta)}\int_0^1 e^{-x/lu}\frac{p_2(u)}{(1+\sigma u\ln\rho)^2+(\pi\sigma u)^2}\frac{du}{\zeta-u}\bigg]. \quad (18)$$

$\varphi(x)$ FOR $\zeta=u_0$

The result for $\varphi(x)$ given by Eq. (18) was obtained subject to $R(\zeta)\geq 0$ but for $\zeta\neq u_0$ and not lying in the interval (0, 1). One may nevertheless obtain the correct result for $\zeta=u_0$ from Eq. (18) by setting $\zeta=u_0+\Delta\zeta$ and canceling the first-order infinities of the first two terms against each other, or one may set $\zeta=u_0$ at the beginning and evaluate $(A+C)$ about the second-order pole then existing at $z=u_0$. Using the first procedure, one has for $\zeta=u_0+\Delta\zeta$

$$e^{-x/l\zeta}=e^{-x/lu_0}[1+x\Delta\zeta/lu_0^2],$$
$$p(\zeta)=1+\sigma(u_0+\Delta\zeta)\ln\frac{u_0-1+\Delta\zeta}{u_0+1+\Delta\zeta}$$
$$=(\Delta\zeta)\frac{1-u_0^2(1-2\sigma)}{u_0(u_2^2-1)}-(\Delta\zeta)^2\frac{2\sigma}{(u_0^2-1)^2}. \quad (19)$$

Also, one may define a function $a(u_0)$ by the relation valid to the first order in $\Delta\zeta$

$$p_2(u_0)/p_2(u_0+\Delta\zeta)=1-\Delta\zeta a(u_0). \quad (20)$$

Introducing these results into Eq. (18) and passing to the limit of $\Delta\zeta=0$, one obtains

$$\varphi(x)=\frac{2\pi n}{\sigma}\bigg[\frac{2\sigma u_0^2}{(1-u_0^2(1-2\sigma))^2}e^{-x/lu_0}$$
$$-\frac{(u_0^2-1)}{1-u_0^2(1-2\sigma)}\left\{1-\frac{x}{lu_0}-u_0a(u_0)\right\}e^{-x/lu_0}$$
$$-\frac{\sigma u_0}{p_2(u_0)}\int_0^1 e^{-x/lu}\frac{p_2(u)}{(1+\sigma u\ln\rho)^2+(\pi\sigma u)^2}\frac{du}{u_0-u}. \quad (21)$$

Since this result is of interest principally for the sake of completeness, the details for the evaluation of $a(u_0)$ will not be given. In order to find this function one must return to the defining equation for $p_2(z)$, introduce the definition of $p(iz)$, which may be expressed in the form

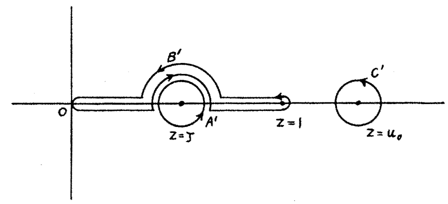

FIG. 4.

$$p(iz)=1-2\sigma z\tan^{-1}(1/z) \quad (22)$$

and then evaluate the logarithm of the desired ratio to the first order. One obtains

$$u_0a(u_0)=\tfrac{1}{2}$$
$$-\frac{1}{\pi}\int_0^\infty\frac{1+u_0^2t^2(1-2\sigma)}{1-2\sigma u_0t\tan^{-1}(1/u_0t)}\frac{dt}{(1+t^2)(1+u_0^2t^2)}.$$

$\varphi(x)$ FOR ζ IN THE INTERVAL (0, 1)

When ζ takes on real values in the open interval (0, 1) the integrand in the definite integral appearing in the expression for $\varphi(x)$ given by Eq. (18) diverges at one point in the range of integration because of the factor $(\zeta-u)$ in the denominator, and the first term of $\varphi(x)$ is complex and indeterminate because of the appearance of $p(\zeta)$, which, according to Eq. (5) and Fig. 1 has the value

$$p(\zeta)=1+\sigma\zeta(\ln\rho\pm i\pi)\quad \rho=(1-\zeta)/(1+\zeta)$$

on this interval. We will show, however, that Eq. (18) still gives $\varphi(x)$ correctly providing that for ζ lying in the open interval (0, 1) the principal part of the integral is taken, and that $p^{-1}(\zeta)$ is interpreted to mean the average value of $p^{-1}(\zeta)$ on the branch cut. Finally, we will show that the point $\zeta=1$ offers no particular difficulty, because the first term of Eq. (18) vanishes at $\zeta=1$ for all directions of approach, and the integral is finite, though the integrand diverges at the upper limit of the integration range. From these interpretations together with those of the preceding section of the terms in

Eq. (18) this form for $\varphi(x)$ may be regarded as correct for all values of ζ with $R(\zeta)\geq 0$.

In order to establish the desired result we return to the evaluation of $(A+B)$ of Eq. (14) with the point $t=-1/l\zeta$ of the t plane lying on the branch cut. This integral may be performed in either of two ways. We may deform the branch cut and evaluate A and B separately as before. In the z plane the contours then have the form given in Fig. 4. The branch cut is inclosed entirely within B'. Now if B' is divided into two parts, the two small semicircles and the straight line portions, one obtains from the latter the same definite integral as before except that the principal part is meant. For the evaluation of A we recall from Eq. (5) and Fig. 1 that with the branch cut having been deformed upward, the contour for A lies everywhere on that branch of $p(z)$ for which α of Eq. (5) has the value $-\pi$ on the real axis. Then A has the same form as given by Eq. (15), but with $p(\zeta)$ given by

$$p(\zeta)=1+\sigma\zeta\{\ln[(1-\zeta)/(1+\zeta)]-i\pi\}.$$

Therefore

$$A=\frac{2\pi e^{-x/l\zeta}}{l\zeta}\frac{1+\sigma\zeta\ln[(1-\zeta)/(1+\zeta)]}{\{1+\sigma\zeta\ln[(1-\zeta)/(1+\zeta)]\}^2+(\pi\sigma\zeta)^2}+\frac{2\pi^2 i\sigma e^{-x/l\zeta}}{l}\left[\left(1+\sigma\zeta\ln\frac{1-\zeta}{1+\zeta}\right)^2+(\pi\sigma\zeta)^2\right]^{-1}.$$

Now we evaluate the parts of B consisting of the two semicircles. Setting $z=\zeta+\epsilon e^{i\theta}$ where ϵ is the radius of the semicircles, taking the limit for $\epsilon\rightarrow 0$, and remembering that the lower and upper semicircles belong to branches of α having the values $-\pi$ and π, respectively, on the real axis, one obtains for this part of B the result

$$-\frac{2\pi^2 i\sigma}{l}e^{-x/l\zeta}\left[\left(1+\sigma\zeta\ln\frac{1-\zeta}{1+\zeta}\right)^2+(\pi\sigma\zeta)^2\right]^{-1}.$$

Thus, this part of B just cancels the imaginary term of A. Now since

$$\frac{1}{2}\left[\frac{1}{p(\zeta+i0)}+\frac{1}{p(\zeta-i0)}\right]=\frac{1+\sigma\zeta\ln[(1-\zeta)/(1+\zeta)]}{\{1+\sigma\zeta\ln[(1-\zeta)/(1+\zeta)]\}^2+(\pi\sigma\zeta)^2} \qquad |\zeta|=\zeta,\quad 0<\zeta<1 \tag{23}$$

one finds that this result for ζ in the interval (0, 1) is the same as before except for the necessary changes in the interpretations of $p^{-1}(\zeta)$ and the definite integral which were asserted above.

A second method of performing the integration for $(A+B)$ when ζ lies in the interval (0, 1) is to leave the branch cut undeformed, and to integrate over the single closed path inclosing the branch cut. This contour may, however, be regarded as two straight lines broken at $z=\zeta$, giving the principal value integral, plus a small circle about $z=\zeta$. In the integration over the circle it must be remembered that α assumes values belonging to different branches on the upper and lower halves of the circle. It is this method which may be used in particular for the point $\zeta=1$. Considering the circle to be of radius ϵ one obtains the same definite integral as before for the straight line portions of the contour except for the upper limit which is now $(1-\epsilon)$. For the integral over the circle one introduces $z=1+\epsilon e^{i\theta}$, and integrates over θ from $-\pi$ to π. This integral is observed to vanish in the limit of ϵ approaching zero. Returning to the definite integral one finds that for small ϵ, the integrand behaves like $\epsilon^{-1}(\ln\epsilon)^{-2}$. The integrand diverges but the integral is easily shown to be finite. This result for $\zeta=1$ may be obtained, also, directly from Eq. (18) in which ζ is permitted to approach unity from any direction other than along the real segment (0, 1). The definite integral remains finite, and the first term of Eq. (18) vanishes because of the unbounded character of $p(\zeta)$ at $\zeta=1$.

Final Results for $F(x)$

The case of incident currents and source distributions will be expressed separately. For an incident current distribution of the form

$$f_0(z)=f\delta(z-\zeta),\quad 0<\zeta\leq 1$$

one obtains from Eqs. (10) and (18) with $n=f\sigma$, and after inserting (23) since ζ is real and in the interval (0, 1)

$$F(x)=2\pi f\left[\frac{1+\sigma\zeta\ln[(1-\zeta)/(1+\zeta)]}{(1+\sigma\zeta\ln[(1-\zeta)/(1+\zeta)])^2+(\pi\sigma\zeta)^2}e^{-x/l\zeta}\right.$$
$$\left.+\frac{\zeta}{u_0-\zeta}\frac{(u_0^2-1)}{1-u_0^2(1-2\sigma)}\frac{p_2(u_0)}{p_2(\zeta)}e^{-x/lu_0}+\frac{\sigma\zeta}{p_2(\zeta)}\int_0^1 e^{-x/lu}\frac{p_2(u)}{(1+\sigma u\ln\rho)^2+(\pi\sigma u)^2}\frac{du}{u-\zeta}\right], \quad (24)$$

where, except for $\zeta=1$ the principal part of the integral is to be taken. For a source distribution of the form

$$q(x)=qe^{-x/l\zeta}\quad R(\zeta)\geq 0$$

one obtains from Eqs. (11) and (18) with $n=ql/v$ and setting $2l_s=l/\sigma$

$$F(x)=\frac{2\pi ql}{\sigma v}\left[\left\{\frac{1}{p(\zeta)}-1\right\}e^{-x/l\zeta}+\frac{\zeta}{u_0-\zeta}\frac{(u_0^2-1)}{1-u_0^2(1-2\sigma)}\frac{p_2(u_0)}{p_2(\zeta)}e^{-x/lu_0}\right.$$
$$\left.+\frac{\sigma\zeta}{p_2(\zeta)}\int_0^1 e^{-x/lu}\frac{p_2(u)}{(1+\sigma u\ln\rho)^2+(\pi\sigma u)^2}\frac{du}{u-\zeta}\right], \quad (25)$$

where, if ζ lies in (0, 1) the right member of Eq. (23) is to be substituted for $p^{-1}(\zeta)$ and the principal part of the integral is to be taken, and if $\zeta=u_0$, the limit of the first two terms leading to the result for $\varphi(x)$ given by Eq. (21) is to be taken. In each of these expressions $\rho=(1-u)/(1+u)$, u_0 must be determined as the positive root of

$$1+\sigma u_0\ln[(u_0-1)/(u_0+1)]=0 \quad (26)$$

and the functions $p_2(\zeta)$ and $p(\zeta)$ are obtained from Eqs. (2) and (4), respectively.

The case of a constant source distribution is of particular interest. The result for constant sources q may be obtained from Eq. (25) by letting $\zeta\to\infty$. One shows easily from the defining equations that

$$\lim_{\zeta\to\infty}p(\zeta)=1-2\sigma,\quad \lim_{\zeta\to\infty}p_2(\zeta)=(1-2\sigma)^{\frac{1}{2}},$$

and obtains, therefore, for $q(x)=q$

$$F(x)=\frac{4\pi ql}{v(1-2\sigma)}\left[1-\frac{(1-2\sigma)^{\frac{1}{2}}}{2\sigma}\frac{(u_0^2-1)}{1-u_0^2(1-2\sigma)}p_2(u_0)e^{-x/lu_0}\right.$$
$$\left.-\tfrac{1}{2}(1-2\sigma)^{\frac{1}{2}}\int_0^1 e^{-x/lu}\frac{p_2(u)}{(1+\sigma u\ln\rho)^2+(\pi\sigma u)^2}du\right]. \quad (27)$$

Defining a capture mean time $\tau_c=l_c/v$ where $1/l=1/l_c+1/l_s$ and $\sigma=l/2l_s$, one obtains for $F(\infty)$ and $F(0)$, the density at points located far from the boundary surface and at the boundary surface, respectively

$$F(\infty)=4\pi q\tau_c, \quad (28)$$

$$\frac{F(0)}{F(\infty)}=1-\tfrac{1}{2}\frac{(1-2\sigma)^{\frac{1}{2}}}{\sigma}\frac{(u_0^2-1)}{1-u_0^2(1-2\sigma)}p_2(u_0)-\tfrac{1}{2}(1-2\sigma)^{\frac{1}{2}}\int_0^1\frac{p_2(u)}{(1+\sigma u\ln\rho)^2+(\pi\sigma u)^2}du. \quad (29)$$

The first result agrees with that obtained from more elementary considerations[5] as it should. The ratio $F(0)/F(\infty)$ depends only on σ, though $F(x)/F(\infty)$ depends on both l and σ.

It is of some interest to compare these results for $F(0)$ with a direct result obtained by integration

[5] H. A. Bethe, Rev. Mod. Phys. 9, 126 (1937).

over angles of $f(0, z)$ given by Eq. (1). In the limiting case of $\sigma=0$ (pure capture) this comparison is easily made. For the case of source distributions both methods give

$$F(0)=2\pi q\tau_c\zeta \ln (1+\zeta^{-1}).$$

The verification of the expressions for $F(0)$ for the limiting case of $\sigma=\frac{1}{2}$ (no capture) by comparison with the direct calculation is not as easily established. In this limit $F(0)$ for a constant finite source distribution ($\zeta=\infty$) is unbounded for $\sigma\rightarrow\frac{1}{2}$. The direct calculation shows that the approach of $F(0)$ to infinity is like $(1-2\sigma)^{-\frac{1}{2}}$. In order to establish this result from Eq. (29) one must first determine the behavior of $p_2(u_0)$ for $\sigma\rightarrow\frac{1}{2}$. One finds that whereas[2]

$$\lim_{z\rightarrow\infty} p_2(z)=1/\sqrt{3}z \quad \sigma=\tfrac{1}{2},$$

σ having passed to the limit $\frac{1}{2}$ before taking the limit in z, the result of a simultaneous approach to their limits through the maintenance of the relation given by (26) is

$$\lim_{u_0\rightarrow\infty} p_2(u_0)=2/\sqrt{3}u_0.$$

Then the second term of (29) may be shown to have a leading term of unity so that the entire expression leads with a term of order $(1-2\sigma)^{\frac{1}{2}}$. Since $\tau_c\sim(1-2\sigma)^{-1}$ the expression for $F(0)$ given by (29) is shown to increase like $(1-2\sigma)^{-\frac{1}{2}}$ for $\sigma\rightarrow\frac{1}{2}$, as it should.

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The Relative Stopping Powers of Carbon and Lead for Slow Mesons

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An experimental determination of the relative stopping powers of carbon and lead for slow mesons has been made with an anticoincidence arrangement of G-M counters. For mesons having an average energy of approximately 4×10^7 ev, the experiments revealed that 28.5 g/cm² of carbon is equivalent in stopping power to 24 ± 5 g/cm² of lead. This gives a value for the ratio of the stopping powers of equal masses of carbon and lead, $S_C/S_{Pb}=0.84\pm0.18$, to be compared with the theoretical value of $S_C/S_{Pb}=1.82$ calculated with the ionization theory. However, it is not necessary to invoke a hitherto unknown absorption process to account for this discrepancy, since it may be explained as arising from the scattering and transition effects. The data corrected for these effects show that 28.5 g/cm² of carbon is equivalent in stopping power to 45 ± 7 g/cm² of lead, and $S_C/S_{Pb}=1.6\pm0.3$, in agreement with the theoretical value within the experimental uncertainty. These experiments indicate that in a dense absorber, any additional absorption process is unimportant compared with ionization for mesons having energies of about 4×10^7 ev, or higher.

INTRODUCTION

EXPERIMENTAL evidence consistent with the possibility that some hitherto unknown process is partially responsible for the removal of slow mesons from the cosmic radiation was obtained during the course of several recent investigations relating to the instability of the meson. Of these the following may be mentioned: (1) The value of τ_0, the proper lifetime of the meson, calculated from the observed rate of production of disintegration electrons is larger than that calculated from the rate of disappearance of mesons.[1,2] This arises from a deficiency in the relative abundance of disintegration

[1] M. A. Pomerantz, Phys. Rev. 57, 3 (1940).
[2] M. Ageno, G. Bernadini, N. Cacciapuoti, B. Ferretti, and G. C. Wick, Phys. Rev. 57, 945 (1940).