Note on Interaction Energies in Radiation Theory

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Radiation interaction energies are calculated for a photon and electron, a photon and scalar or vector mesotron, two electrons, and two scalar mesotrons. For a charged particle and photon the interaction energy is inversely proportional to the frequency of the photon and does not transform like the time component of a four-vector. For two charged particles the interaction energy is closely related to the Møller matrix element.

I. INTRODUCTION

THE procedure in calculating the energy eigenvalues of a system consisting of material particles and a radiation field is to solve the problem exactly with the matter and radiation considered as independent and then treat the interaction as a perturbation. But the change in energy levels due to the perturbation turns out to be infinite so that it is not possible to obtain a satisfactory theory of the self-energy of a single particle by this method. As is well known, however, those results of quantum electrodynamics which are concerned with the interaction of two or more particles are found to be in good agreement with experiment. It is therefore of interest to see whether the ordinary technique of self-energy calculations can be made to give sensible results when applied to the interaction energy of several particles. To obtain this interaction energy it is necessary to calculate the energy when all the particles are present together and subtract from this the energy of each of the particles when it is present by itself. All these energies are infinite, but if the subtractions are made in the most natural manner the answers are definite. Actually, in a theory in which pair production is possible a subtraction of this kind is necessary even to obtain the self-energy of a single particle since the perturbed energy is infinite even for a vacuum.

In this paper interaction energies are calculated for a photon interacting with an electron, scalar mesotron, or vector mesotron; two electrons, and two scalar mesotrons. In all cases the results are finite in the order e^2 and may be given simple physical interpretations. The interaction energy of a photon and electron or mesotron is inversely proportional to the frequency of the photon, and does not transform like the time component of a four-vector. The reason for the latter is that it is not possible to define an energy-momentum four-vector for a system consisting of charged particles and a radiation field without taking adequate account of the forces holding the charges together. The divergent self-energy results show that this is not done adequately in the present quantum theory.

In finding the energy level perturbations due to the interaction between matter and radiation field either of two procedures may be used. One is to apply the ordinary methods of perturbation theory directly.[1] The other, developed by Weisskopf, is to express the change in energy levels as the average value of certain operators over the perturbed state.[2,3] The latter method gives more insight into the physical significance of the various terms. In this paper both procedures will be described.

II. INTERACTION OF A PHOTON AND ELECTRON

The Hamiltonian of the system consisting of electrons and radiation field may be written

$$\mathcal{H}=\frac{1}{8\pi}\int(E^2+H^2)d\mathbf{r}-\frac{1}{4\pi}\int\varphi\cdot\operatorname{div}E d\mathbf{r}$$
$$+\int\left(\rho\varphi-\frac{\mathbf{i}\cdot\mathbf{A}}{c}\right)d\mathbf{r}+c\int\psi^*(\boldsymbol{\alpha}\cdot\mathbf{p}+\beta mc)\psi d\mathbf{r}, \quad (1)$$

where the third integral constitutes the interaction since it contains variables of both matter

[1] I. Waller, Zeits. f. Physik **62**, 673 (1930).
[2] V. Weisskopf, Zeits. f. Physik **89**, 27 (1934); **90**, 817 (1934).
[3] V. Weisskopf, Phys. Rev. **56**, 72 (1939).

and field. Here

$$\rho = e\psi^*\psi - \sigma, \quad \mathbf{i}/c = e\psi^*\boldsymbol{\alpha}\psi, \tag{2}$$

where e is the algebraic value of the electron charge, and σ is the charge density of electrons in negative energy states. Using second quantization we write

$$\psi = \sum a_q \varphi_q,$$
$$\varphi_q = (u_q/V^{\frac{1}{2}}) \exp\left[i(\mathbf{p}_q \cdot \mathbf{r})/\hbar\right],$$

where u_q is a normalized spinor. The a_q's are the Fermi-Dirac operators satisfying the relations

$$a_q a_{q'}^* + a_{q'}^* a_q = \delta_{qq'},$$
$$a_q a_{q'} + a_{q'} a_q = 0,$$
$$a_q^* a_{q'}^* + a_{q'}^* a_q = 0.$$

By eliminating the longitudinal field components the Hamiltonian may be put in the form

$$\mathfrak{H} = H_r + H_m + H',$$

$$H' = W_{st} - \int \frac{\mathbf{i}\cdot\mathbf{A}_t}{c} d\mathbf{r},$$

$$W_{st} = \frac{1}{2}\int \frac{\rho(\mathbf{r}_1)\rho(\mathbf{r}_2)}{|\mathbf{r}_1 - \mathbf{r}_2|} d\mathbf{r}_1 d\mathbf{r}_2,$$

$$H_m = c\int \psi^*(\boldsymbol{\alpha}\cdot\mathbf{p} + \beta mc)\psi d\mathbf{r} = \sum N_q E_q,$$

$$H_r = \sum M_s h\nu_s.$$

$\mathbf{A}_t$ is the transverse part of the vector potential and is given by

$$\mathbf{A}_t = \sum_s \mathbf{e}_s \frac{ch^{\frac{1}{2}}}{(V\nu_s)^{\frac{1}{2}}}\left\{A_s^* \exp\left[-\frac{i\mathbf{k}_s\cdot\mathbf{r}}{\hbar}\right] + A_s \exp\left[\frac{i\mathbf{k}_s\cdot\mathbf{r}}{\hbar}\right]\right\}.$$

$\mathbf{k}_s$ is the momentum of the photon s which is polarized in the direction $\mathbf{e}_s$ and has frequency $\nu_s = ck_s/h$. The A_s's are the Einstein-Bose operators of the radiation field and are defined as follows:

$$(M_s|A_s|M_s+1) = (M_{s+1}|A^*|M_s) = (M_s+1)^{\frac{1}{2}} \tag{3}$$

and all other elements are zero. We have taken a gauge in which the scalar potential of the radiation field vanishes.

If $\Phi(\cdots M_s \cdots; \cdots N_q \cdots)$ is the probability amplitude for the state in which M_s photons are present in state s, and N_q electrons are present in state q, we have

$$i\hbar\partial\Phi/\partial t = (H_m + H_r + H')\Phi.$$

We try to obtain solutions of this equation in the form

$$\Phi = e^{-iWt/\hbar}(\Phi_0 + \lambda\Phi_1 + \cdots),$$

where λ is proportional to the electron charge, and

$$\Phi_0(\cdots M_s \cdots; \cdots N_q \cdots) = \cdots \overset{M_s}{\delta_{M_s^0}} \cdots \overset{N_s}{\delta_{N_s^0}} \cdots$$

so that

$$(H_m + H_r)\Phi_0 = W_0\Phi_0.$$

Then

$$W = W_0 + (o|H'|o) + \sum_j \frac{|(j|H'|o)|^2}{W_0 - W_j} \tag{4}$$

where o and j are abbreviations for complete sets of quantum numbers. The only part of H' which has diagonal elements is W_{st} and this does not contribute to the interaction of a photon and electron. Therefore to compare the energy of two states we simply compare terms in the sum over j. We then have for the interaction energy of a photon and electron

$$\Delta W = (W_{\text{vac+el+phot}} - W_{\text{vac}}) - (W_{\text{vac+el}} - W_{\text{vac}}) - (W_{\text{vac+phot}} - W_{\text{vac}}).$$

The calculations are slightly different depending on whether or not the negative energy electron states are assumed occupied. If we assume them empty we have

$$\Delta W = (A) + (B) + (C) + (D),$$

where (A), (B), (C), and (D) are the terms appearing in the sum over j for the following virtual transitions:

(A): Absorption of the photon; the electron goes to a positive energy state.

(B): Stimulated emission of a second photon similar to the one already present; the electron goes to a positive energy state.

(C) and (D): Same as (A) and (B) with the electron going to a negative energy state. For the

negative energy states assumed filled we have

$$\Delta W=(A)+(B)-(C')-(D'),$$

where (A) and (B) are the same as before, and (C') and (D') are described as follows:

(C'): In calculating $W_{\text{vac+el+phot}}$ the transition in which a vacuum electron absorbs the photon and goes to the state occupied by the positive energy electron does not appear because of the exclusion principle.

(D'): Similarly a vacuum electron cannot make a transition to the state occupied by the positive energy electron with stimulated emission of the photon.

Writing out the matrix elements and energy denominators explicitly and remembering that in all cases the spins of either the initial or final states of the virtual transitions are arbitrary we find, whether the negative energy states are occupied or not,

$$\Delta W=\frac{2\pi e^2ch^2}{kV}\sum_{l=1}^{4}\left\{\frac{|\{u_l^*(\mathbf{p}_0-\mathbf{k})\alpha_s u_{l_0}(\mathbf{p}_0)\}|^2}{E_{l_0}(\mathbf{p}_0)-E_l(\mathbf{p}_0-\mathbf{k})-ck}+\frac{|\{u_l^*(\mathbf{p}_0+\mathbf{k})\alpha_s u_{l_0}(\mathbf{p}_0)\}|^2}{E_{l_0}(\mathbf{p}_0)-E_l(\mathbf{p}_0+\mathbf{k})+ck}\right\}.$$

Here α_s is the component of $\boldsymbol{\alpha}$ parallel to the polarization of the photon $\mathbf{k}$ and

$$E_{1,2}(p)=+(c^2p^2+m^2c^4)^{\frac{1}{2}},$$

$$E_{3,4}(p)=-(c^2p^2+m^2c^4)^{\frac{1}{2}}.$$

To evaluate the above expression it is necessary to consider the sums over $l=1, 2$ and $l=3, 4$ separately because the energy denominators are different in the two cases. These summations may be performed by means of selection operators in the usual way. It is then seen that the expression is independent of the spin state of the electron so that it is permissible to average over any two independent states by further selection operators. Straightforward computations give

$$\Delta W=\frac{2\pi e^2ch^2}{VkE(p_0)}=\frac{e^2}{2VE(p_0)}\left\langle\int\mathbf{A}_s^2d\mathbf{r}\right\rangle_{\text{Av}},\quad(5)$$

where $\mathbf{A}_s$ is the vector potential due to the photon. If more than one photon is present the interaction energy is just the sum of the interaction energies between the electron and each photon. This follows because from (3) the squares of the matrix elements depend on the number of photons present in the state. (The photons interact with each other only in higher orders than we are considering here.) Therefore if there are M photons of frequency ν per unit volume,

$$\Delta W=\frac{MV2\pi e^2ch^2}{VkE(p_0)}=\frac{1}{2\pi}\frac{e^2c^2}{\nu^2}\frac{G_r}{E(p_0)},$$

where G_r is the radiation energy density.

To obtain this result by the alternative method of Weisskopf we proceed in the following way. It has been shown that the change in energy eigenvalues due to the perturbing Hamiltonian may be obtained by dividing those terms in the perturbation which are explicitly proportional to e by two, leaving the terms explicitly proportional to e^2 unchanged, and then averaging the result over the perturbed state.[4] Therefore

$$W-W_0=\left\langle W_{st}-\frac{1}{2}\int\frac{\mathbf{i}\cdot\mathbf{A}_t}{c}d\mathbf{r}\right\rangle_{\text{Av}}.$$

As before W_{st} does not contribute for a photon and electron. It suffices to replace $\mathbf{A}_t$ by $\mathbf{A}_s$, the vector potential of the photon, and $\mathbf{i}$ by $\mathbf{i}'$, the perturbed current, since the scalar product of the unperturbed current and $\mathbf{A}_s$ averages to zero. Therefore

$$\Delta W=-\left\langle\int\frac{\mathbf{i}'\cdot\mathbf{A}_s}{2c}d\mathbf{r}\right\rangle_{\text{Av}}.\quad(6)$$

We do not include the perturbed current due to the vacuum electrons in $\mathbf{i}'$. The perturbed current may be calculated by considering the radiation field as an externally applied time dependent perturbation. Weisskopf has shown that for a stationary electron

$$\mathbf{i}'=-ce^2\mathbf{A}_s/mc^2V.\quad(7)$$

This is true in either one electron theory or hole theory. To obtain the value of $\mathbf{i}'\cdot\mathbf{A}$ for an electron with arbitrary velocity we make a Lorentz transformation. To carry this through we make use of some simple relations. First it may be shown that for a plane light wave

[4] Reference 3, p. 80.

G_r/ν^2 is an invariant. It then follows that if we always use a gauge in which the scalar potential is zero, $\mathbf{A}^2$ is an invariant. From this result, together with (7) we can prove that $\mathbf{i}'\cdot\mathbf{A}$ is also an invariant. Therefore we have

$$\mathbf{i}'\cdot\mathbf{A}=\mathbf{i}_0'\cdot\mathbf{A}_0=-\frac{ce^2}{V_0}\frac{\mathbf{A}_0^2}{mc^2}=-\frac{ce^2\mathbf{A}^2}{V_0mc^2},$$

where the subscript zero indicates the rest system of the electron. But $V=V_0(1-\beta^2)^{\frac{1}{2}}$ where V and V_0 are the volumes ascribed to the electron in the two systems by the quantization procedure, and $\beta=v/c$. Therefore

$$\mathbf{i}'\cdot\mathbf{A}_s=\frac{-ce^2\mathbf{A}_s^2}{Vmc^2}(1-\beta^2)^{\frac{1}{2}}=-\frac{ce^2A_s^2}{VE(p_0)}. \quad (8)^5$$

This expression may also be obtained by carrying through the original calculations for general velocity of the electron. Inserting (8) into (6) we obtain (5) again. Since G_r/ν^2 is an invariant, ΔW transforms as $1/E(p_0)$ and is not the time component of a four-vector.

III. Mesotron and Photon

The calculations for mesotrons are essentially the same as for the Dirac electron. But here the interaction Hamiltonians have terms proportional to $e^2\mathbf{A}^2$ which give rise to diagonal elements that contribute to the expression (4).[6] The off-diagonal elements of the part of the interaction proportional to $e\mathbf{A}$ again give contributions proportional to e^2A^2 through the last term in (4). The virtual transitions occurring are of the same sort as before with transitions involving pair production replacing those described as involving negative energy states. Calculations show that the contributions of the various transitions to the interaction energy always just cancel for a scalar mesotron so that only the diagonal elements remain. For a moving vector mesotron this is in general not so. If we use Weisskopf's alternative method we find that (6) again holds.[6,7] This is because the interaction Hamiltonian for a scalar or vector mesotron is of the form

$$-\int\left(\frac{\mathbf{i}_1\cdot\mathbf{A}}{c}+\frac{\mathbf{i}_2\cdot\mathbf{A}}{2c}\right)d\mathbf{r},$$

where $\mathbf{i}_1$ is the part of the current explicitly independent of $\mathbf{A}$ and proportional to e while $\mathbf{i}_2$ is explicitly proportional to $e^2\mathbf{A}$. To calculate $\mathbf{i}'\cdot\mathbf{A}$ we may again apply a relativity transformation to (7) which holds also for mesotrons.[3,7] By (6) and (8) we see that the interaction energy is again given by (5). It is easy to check the expression (5) by direct calculation for a scalar mesotron with arbitrary velocity, and for a few of the simpler cases for a vector mesotron. For a scalar mesotron the calculations were also carried through with the unquantized Gordon-Klein equation. Equation (5) was again obtained.

IV. Two Charged Particles

To calculate the interaction energy of two electrons it is convenient to start with the Hamiltonian in the form (1). Then, using Weisskopf's method, we have

$$W-W_0=\left\langle\frac{1}{2}\int\left(\rho\varphi-\frac{\mathbf{i}\cdot\mathbf{A}}{c}\right)d\mathbf{r}\right\rangle_{\text{Av}}.$$

We may take

$$\mathbf{A}(\mathbf{r},t)=\int\frac{\mathbf{i}(\mathbf{r}',t-|\mathbf{r}-\mathbf{r}'|/c)}{c|\mathbf{r}-\mathbf{r}'|}d\mathbf{r}',$$

$$\varphi(\mathbf{r},t)=\int\frac{\rho(\mathbf{r}',t-|\mathbf{r}-\mathbf{r}'|/c)}{|\mathbf{r}-\mathbf{r}'|}d\mathbf{r}',$$

where $\mathbf{i}(\mathbf{r},t)$ and $\varphi(\mathbf{r},t)$ are given by (2). These expressions may be evaluated and we find

$$\Delta W=(W_{1+2}-W_{\text{vac}})-(W_1-W_{\text{vac}})-(W_2-W_{\text{vac}})$$

$$=\int\frac{\rho(\mathbf{r}_1)\rho(\mathbf{r}_2)}{|\mathbf{r}_1-\mathbf{r}_2|}d\mathbf{r}_1d\mathbf{r}_2-\int\frac{\mathbf{i}(\mathbf{r}_1)\cdot\mathbf{i}(\mathbf{r}_2)}{c^2|\mathbf{r}_1-\mathbf{r}_2|}d\mathbf{r}_1d\mathbf{r}_2$$

$$-\frac{4\pi e^2\hbar^2c^2}{V}\left[\frac{|\{u_{q_1}^*u_{q_2}\}|^2-\{u_{q_1}^*\boldsymbol{\alpha}u_{q_2}\}\{u_{q_2}^*\boldsymbol{\alpha}u_{q_1}\}}{c^2(\mathbf{p}_1-\mathbf{p}_2)^2-(E_1-E_2)^2}\right],$$

[5] This result is not quite the same as for a classical point charge under the influence of a radiation field. There if the point charge is moving parallel to the electric field we obtain, because of the longitudinal mass,

$$i'\cdot\mathbf{A}_s=-e^2\mathbf{A}_s^2(1-\beta^2)^{\frac{1}{2}}/mc.$$

But if in the rest system there are N_0 charges per unit volume we obtain for the perturbed current per unit volume

$$i'\cdot\mathbf{A}_s=-e^2N\mathbf{A}_s^2(1-\beta^2)^{\frac{1}{2}}/mc.$$

which is analogous to (8). Here N is the number of charges per unit volume in the system in which the charges are moving, and $N=N_0/(1-\beta^2)^{\frac{1}{2}}$. The difference between the two results is caused by fluctuations in N due to the small oscillations of the charges. This causes the term $Ne\mathbf{v}$ in the current density to fluctuate ($\mathbf{v}$ is the unperturbed velocity).

[6] W. Pauli and V. Weisskopf, Helv. Phys. Acta **7**, 709; H. Yukawa, Proc. Phys. Math. Soc. Japan **20**, 319 (1938); N. Kemmer, Proc. Roy. Soc. **A166**, 127 (1938).

[7] R. Richtmyer, Phys. Rev. **57**, 413 (1940).

where $\rho(r_i)$ and $\mathbf{i}(r_i)$ are the average charge and current densities of the two electrons. The first two terms correspond to the interaction of two uniform charge and current distributions over the volume V. The other term comprises the correction to this due to the antisymmetrical character of the wave functions, and the spin motion of the electrons. It has the form of the Møller matrix element of a transition in which the electrons just change places.[8] Of course such a transition has no physical significance. Formally it is forbidden by the exclusion principle and so appears in the interaction energy with a minus sign. If we eliminate the longitudinal field components and use ordinary perturbation theory we obtain the same expression for ΔW with the exception of the second term. This does not appear since in the expansion of the radiation field which is used there is no photon of zero wave-length to enable the constant current distributions to interact.

For two scalar mesotrons the situation is the same except insofar as it is modified by the Einstein-Bose statistics of the particles. In calculating $\mathbf{A}(\mathbf{r})$ and $\varphi(\mathbf{r})$ care must be taken to give the operators appearing in the Fourier expansion of the wave functions their correct time dependence. We find

$$\Delta W=\int\frac{\rho(\mathbf{r}_1)\rho(\mathbf{r}_2)}{|\mathbf{r}_1-\mathbf{r}_2|}d\mathbf{r}_1d\mathbf{r}_2 - \int\frac{\mathbf{i}(\mathbf{r}_1)\cdot\mathbf{i}(\mathbf{r}_2)}{c^2|\mathbf{r}_1-\mathbf{r}_2|}d\mathbf{r}_1d\mathbf{r}_2+M,$$

where[9]

$$M=\frac{\pi e^2\hbar^2c^2[(E(p_1)+E(p_2))^2-c^2(\mathbf{p}_1+\mathbf{p}_2)^2]}{VE(p_1)E(p_2)[c^2(\mathbf{p}_1-\mathbf{p}_2)^2-(E(p_1)-E(p_2))^2]}$$

for like particles, and

$$M=\frac{\pi e^2\hbar^2c^2[(E(p_1)-E(p_2))^2-c^2(\mathbf{p}_1-\mathbf{p}_2)^2]}{VE(p_1)E(p_2)[c^2(\mathbf{p}_1-\mathbf{p}_2)^2-(E(p_1)-E(p_2))^2]}$$

for unlike particles. $\mathbf{p}_i$ is the momentum of the particle.

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[8] C. Møller, Zeits. f. Physik **70**, 790 (1931).

[9] The expression M is analogous to the Møller matrix element for two scalar mesotrons. It has the same transformation properties as the formula obtained for the Dirac electron. (For the Dirac electron

$$|\{u^*_{q_1}u_{q_2}\}|^2-\{u^*_{q_1}\boldsymbol{\alpha}u_{q_2}\}\{u^*_{q_2}\boldsymbol{\alpha}u_{q_1}\}$$

is not an invariant since the u_q's must be taken as normalized in each system and so do not transform in the usual way.)