concepts of the interaction between electrons and nuclei in the elementary scattering process. In addition, we wish to emphasize that the theoretical predictions for secondary production, a process intimately connected with scattering, have been found to be in satisfactory accord with experiment.

In conclusion we wish to express our appreciation to Professors E. J. Williams, J. R. Oppenheimer, and J. A. Wheeler for correspondence and discussions concerning the theoretical aspects of the scattering problem. Our indebtedness to Professor C. C. Lauritsen is also to be acknowledged.

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Scattering and Polarization of Electrons

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The possibility of accounting for the small polarization of electrons, as observed in the double scattering experiments and the anomalously small scattering of fast electrons ($E \gtrsim 500$ kev) in heavy scattering materials, by the assumption of non-Coulombian forces near the nucleus is investigated. It is assumed that the range of the anomalous forces is of the order of the nuclear (or electron) radius so that for all energies of interest ($E < 2$ Mev), the range is much smaller than the wave-length of the electrons. Consequently, the scattering of only $s_{\frac{1}{2}}$ and $p_{\frac{1}{2}}$ electrons need be considered. Without any further assumptions as to the nature of the forces it is found that the phase shifts due to the deviation from the pure Coulomb field are too small to account for the observed scattering and asymmetry (relative difference between scattering at azimuth 0 and π in double scattering) unless the inside wave functions at the boundary are nearly equal to the irregular part of the outside wave functions. In general this can be the case for either the $s_{\frac{1}{2}}$ or the $p_{\frac{1}{2}}$ wave function so that either the $s_{\frac{1}{2}}$ or the $p_{\frac{1}{2}}$ waves are scattered anomalously with appreciable phase shifts, but not both. The results for the scattering and asymmetry, at 90° in Au, when only one wave is scattered are: For the scattering of $s_{\frac{1}{2}}$ waves the asymmetry has a minimum value which for low energies, 100 to 300 kev, is 5 to 6 times greater than the observations allow; at higher energies, 500 to 1500 kev the scattering intensity has a minimum value of 25 to 40 percent of the Coulomb scattering which is somewhat larger than the observed ratio but is perhaps not beyond the limits of experimental error. The asymmetry corresponding to this minimum scattering is about the same as the asymmetry for the Coulomb field, *viz.* about 5 percent at these energies. Without any measurements at high energies it is difficult to exclude the possibility that the asymmetry may be as large as this. When only $p_{\frac{1}{2}}$ waves are scattered anomalously the correct asymmetry and scattering may be obtained at low energies. At high energies the minimum scattering is 70 to 80 percent of the Coulomb scattering which seems much too large. Therefore in order to obtain an asymmetry and scattering which are not in obvious disagreement with the observations it is necessary to assume either a large range or a specialized form for the non-Coulombian forces. The specialized form must be such that either of the following may take place: (1) At low energies only the $p_{\frac{1}{2}}$ wave is scattered and at high energies only the $s_{\frac{1}{2}}$ wave is scattered. (2) Both waves are scattered at all energies despite the small range of the forces. In the first case it seems necessary to postulate an interaction which has a rather strong energy dependence in the energy region where the scattering begins to depart from the Coulombian value. In the second case it is seen that the interaction must be very large, several times $137mc^2$, and in addition a rather special energy dependence would seem necessary. Insofar as these possibilities do not seem plausible it would appear that the anomalous scattering and asymmetry must be explained on grounds other than the existence of non-Coulombian forces.

Introduction

It has been shown by Mott[1] on the basis of the relativistic wave equation of Dirac that an initially unpolarized beam of electrons becomes polarized after a large angle scattering by a heavy nucleus. Moreover, according to the theory the polarization should effect an asymmetry about the azimuth if the beam is scattered a second time from a similarly heavy scattering material through a similarly large scattering angle. It is supposed that the effect be observed under conditions insuring single scattering.

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[1] N. F. Mott, Proc. Roy. Soc. **A135**, 429 (1932).

The intensity of the doubly scattered beam will, for any scattering field, be proportional to $1+\delta^2 \cos \varphi$ where φ is the azimuthal angle, (angle between the two scattering planes). The quantity δ^2 depends on the electron energy and, of course, on the scattering field. The maximum effect, relative difference between scattering at $\varphi=0$ and $\varphi=\pi$, is then $2\delta^2$ which quantity is referred to as the asymmetry of the doubly scattered beam. For scattering in Au and with the assumption of a pure Coulomb field and for both scattering angles equal to 90°, Mott's calculations show an asymmetry which reaches a rather flat maximum of 16 percent at 140 kv energy. With these scattering angles and at energies for which 13 to 16 percent asymmetry would be expected the experiments show that within the experimental error of one percent there is no asymmetry in the scattering in Au.[2]

For the scattering angles in question a deviation from a pure Coulomb field due to screening by the atomic electrons can hardly be expected to be effective. This is in fact shown by the work of Sauter.[3] An attempt to explain the discrepancy as due to the depolarizing action of (1) multiple scattering, (2) inelastic scattering with spin change of the incident electron and/or (3) exchange scattering in which the exchanged electrons have opposite spins gave entirely negative results.[4] Of these three effects the first is the most important and with the largest foil thickness used in the experiments a reduction of the asymmetry by at most 2 percent of the Mott value can be expected. Since all other ***known*** perturbations which may act in the scattering process are very small compared to the Coulomb field it does not seem fruitful to look in this direction for an appreciable modification of the theoretical results.

In addition to an anomalous polarization and asymmetry, the measurements of the elastic scattering of fast electrons by heavy and medium heavy nuclei also show abnormal effects for the absolute intensity. Up to energies of about 200 kev there is evidence that the scattering in both heavy and light elements is in accord with the expectations based on a pure Coulomb field.[5] For heavy scatterers, in which we are mainly interested, measurements by Dymond[2] of the 90° scattering in Au show agreement within a few percent with the theoretical value for energies up to 160 kev. The energy region in which the scattering is at least approximately Coulombian no doubt extends to somewhat higher energies (say 300 to 400 kev) as is suggested by the measurements of Chadwick and Mercier.[6] However, present evidence is that the scattering is sharply reduced for energies greater than about 500 kev and for scatterers at least as light as Kr. Thus Klarman and Bothe[7] have observed only one-fifth the Mott scattering in Kr and Xe for β-particles of energies between 500 kev and 2.5 Mev scattered at angles greater than 40°. The scattering of 400-kev−1-Mev β-particles in Hg according to Barber and Champion[8] is only about one-sixth the Mott scattering, the factor varying only slightly for different scattering angles. In rough agreement with the results of Klarman and Bothe for the scattering in Xe, Barber and Champion[9] have recently reported an anomalous scattering of 700-kev−1.2-Mev β-particles in I which they found to be about 40 percent of the Mott scattering.[10, 11]

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[2] E. G. Dymond, Proc. Roy. Soc. **A136**, 638 (1932); **A145**, 657 (1934). For further references and a critical survey of the experimental work see H. Richter, Ann. d. Physik **28**, 533 (1937). Recently K. Kikuchi, Proc. Phys-Math. Soc. of Japan **12**, 524 (1939) has observed an asymmetry in Au at 75 kev which is in apparent agreement with the theoretical value of 10 percent. Actually, this experiment, if correct, would indicate an asymmetry much *larger* than theory predicts because with the thick (2×10^{-3} cm) targets used the multiple scattering will result in a large depolarization—about 80 percent (cf. reference 4 below). However, while no information was given as to precautions taken against stray electrons, x-rays, etc., it is almost certain that the instrumental asymmetry was not properly taken into account. As Dymond (cf. above) has shown, the instrumental asymmetry due to nonideal geometry can easily be as large as the theoretical asymmetry so that the whole of the Kikuchi effect might be accounted for in this way.

[3] F. Sauter, Ann. d. Physik **18**, 61 (1933).

[4] M. E. Rose and H. A. Bethe, Phys. Rev. **55**, 277 (1939).

[5] We exclude from our consideration small angle scattering where screening or multiple scattering may become of importance.

[6] J. Chadwick and P. H. Mercier, Phil. Mag. **50**, 208 (1925).

[7] H. Klarman and W. Bothe, Zeits. f. Physik **101**, 489 (1936).

[8] A. Barber and F. C. Champion, Proc. Roy. Soc. **A168**, 159 (1938).

[9] F. C. Champion and A. Barber, Phys. Rev. **55**, 111 (1939).

[10] The Mott scattering with which these authors have compared their results is the Rutherford scattering with relativistic corrections but with only first-order corrections in the fine structure constant (times atomic number) taken into account. However, the difference between this and the rigorous expression (see reference 1 and J. H. Bartlett

The existence of these two anomalies, small asymmetry and diminished scattering at high energies, has led to some speculation in regard to a possible failure of the Dirac theory. Such an eventuality does not seem plausible in the energy region under consideration and in the present paper an appeal will be made to a breakdown of the Coulomb field at small distances as a more plausible source of the difficulties. It is essential to note that the asymmetry and the scattering are not independent of each other but both depend on the same pair of scattering amplitudes (see next section). Thus, while it is evident that it is possible to introduce a deviation from the Coulomb field which will result in a small polarization and asymmetry of the twice-scattered beam,[12] it is not at all evident that the same non-Coulombian forces will account for the observed scattering as well. It is obvious that in general such will not be the case. Our purpose here is therefore to determine whether there exists a non-Coulombian interaction at small distances which is consistent with both small asymmetry and diminished scattering at large energies.

Scattering and Polarization

In the following no assumptions will be made other than the existence of a non-Coulombian field of short range. Beyond a distance of the order of the electron (or nuclear) radius the field is to be pure Coulombian. Before we investigate the scattering and polarization in such a field it will first be necessary to recapitulate some of the formalism developed by Mott[1] in the form applicable to scattering in fields which are Coulombian at large distances.

General formalism

The elastic scattering of electrons incident along the z axis with wave number k and scattered in the direction ϑ, φ by a force field which is Coulombian at large distances may be described by the asymptotic form of the third and fourth components of the Dirac wave function:

$$\psi_\nu \sim p_\nu e^{ikz} + S_\nu e^{i(kr+a\log 2kr)}/kr, \quad a = e^2Z/\hbar v, \quad \nu = 3, 4. \tag{1}$$

Here p_ν are the components of the amplitude of the Dirac plane wave. In particular

$$p_3 = A, \qquad p_4 = B, \tag{2}$$

where A and B are constants related to the direction of the spin axis of the incident electrons and subject to the normalization condition

$$|A|^2 + |B|^2 = 1. \tag{3}$$

The general solution of the scattering problem may be obtained by a superposition of the two waves corresponding to $A=1$, $B=0$ and $A=0$, $B=1$. Mott has shown that in these two cases the asymptotic forms of ψ_3 and ψ_4 are given by

$$\begin{aligned} \psi_3 &\sim e^{ikz} + f(\vartheta)e^{i(kr+a\log 2kr)}/kr, \\ \psi_4 &\sim g(\vartheta)e^{i\varphi}e^{i(kr+a\log 2kr)}/kr, \end{aligned} \tag{4a}$$

for $A=1$, $B=0$ and

$$\begin{aligned} \psi_3 &\sim -g(\vartheta)e^{-i\varphi}e^{i(kr+a\log 2kr)}/kr, \\ \psi_4 &\sim e^{ikz} + f(\vartheta)e^{i(kr+a\log 2kr)}/kr, \end{aligned} \tag{4b}$$

for $A=0$, $B=1$. It is essential to note that the scattering amplitudes f and g, depending on ϑ alone, are the same functions in both cases.

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and R. E. Watson, Phys. Rev. **56**, 612 (1939)), is of no great importance in view of the magnitude of the experimental error involved in the measurements.

[11] In contrast to the results described above Stepanowa, Physik. Zeits. Sowjetunion **12**, 550 (1937) finds the scattering of 200-kev–1.1-Mev β-particles in N to be 1.5 times the expected scattering and for 1.5–3 Mev 30 times the theoretical scattering. These results are in contradiction with the measurements of Champion, Proc. Roy. Soc. **A153**, 353 (1936) who found the scattering of 400-kev–1.1-Mev β-particles in N to be normal. Although this discrepancy has not been resolved and further measurements are desirable, we shall proceed on the assumption that the scattering should be in accord with the results of Champion *et al.*

[12] It has been shown by O. Halpern and J. Schwinger, Phys. Rev. **48**, 109 (1935) that the asymmetry can be made small by assuming a repulsion at a distance of the order of the nuclear radius. In a qualitative way this may be understood as an interference effect in which the polarization arising from the spin-orbit interaction in the scattering field at large distances is canceled by the contribution of that arising from small distances. The result of Halpern and Schwinger was based on the use of the Born approximation which is admittedly a poor one for the polarization. For example, for the Coulomb field the Born approximation underestimates the asymmetry by a factor of 6 or so, cf. F. Sauter, reference 3.

Then the scattering cross section per unit solid angle is

$$\Phi = I/k^2 = (|f|^2+|g|^2)/k^2, \tag{5}$$

and the asymmetry is given by

$$2\delta^2 = 2[i(fg^*-f^*g)]^2/I^2. \tag{6}$$

Since we wish to find the scattering amplitudes f and g it is sufficient to consider the case $A=1$, $B=0$ so that we need consider only the asymptotic forms (4a). It has been shown by Mott that the wave functions with the correct asymptotic behavior are

$$\begin{aligned}\psi_3 &= i\sum_0^\infty[(l+1)e^{i\eta_l}G_l + le^{i\eta_{-l-1}}G_{-l-1}](-)^l P_l,\\ \psi_4 &= i\sum_0^\infty[-e^{i\eta_l}G_l + e^{i\eta_{-l-1}}G_{-l-1}](-)^l P_l^1 e^{i\varphi},\\ P_l^1 &= \sin\vartheta(d/d\cos\vartheta)P_l.\end{aligned} \tag{7}$$

Here G_l and G_{-l-1} are the radial functions of the third and fourth Dirac components for $j=l+\frac{1}{2}$ and $j=l-\frac{1}{2}$, respectively. These are chosen to have the asymptotic form at large r:

$$G_\kappa \sim (kr)^{-1}\cos(kr+a\log 2kr+\eta_\kappa), \tag{8}$$

where for the Coulomb field the phase η_κ is given by

$$e^{2i\eta_\kappa} = e^{-\pi i\gamma_\kappa}\frac{\kappa+1+ib}{\gamma_\kappa+ia}\,\frac{\Gamma(\gamma_\kappa-ia)}{\Gamma(\gamma_\kappa+ia)}, \tag{9}$$

$$\gamma_\kappa = ((\kappa+1)^2-\alpha^2Z^2), \quad b = a(1-v^2/c^2)^{\frac{1}{2}}, \tag{9a}$$

and we have introduced

$$\begin{aligned}\kappa &= l && \text{for} \quad j=l+\tfrac{1}{2}, \quad l\geqslant 0,\\ &= -l-1 && \text{for} \quad j=l-\tfrac{1}{2}, \quad l\geqslant 1.\end{aligned} \tag{9b}$$

Thus $\kappa=0$ and -2 for the $s_{\frac{1}{2}}$ and $p_{\frac{1}{2}}$ waves, respectively.

From (7) and (8) the asymptotic form of ψ_3 and ψ_4 may be found and a comparison with (4a) gives the amplitudes f and g.

Phase shifts in the non-Coulombian field

We now consider that the scattering field is Coulombian only for distances larger than some radius r_0 and inside r_0 has some other behavior which need not be specified. The only restriction we shall impose on the deviation from the Coulomb field is that the range r_0 shall be much smaller than all wave-lengths in which we shall be interested. For a range of order of the nuclear radius this condition will be well fulfilled for even the most energetic particles used in the scattering experiments (wave-length $\sim\hbar/mc$). It then follows that the only waves which are appreciably perturbed by the anomalous potential are the $s_{\frac{1}{2}}$ and $p_{\frac{1}{2}}$ waves both of which have the same total angular momentum, $j=\frac{1}{2}$. Moreover, since we shall be interested only in scattering angles of $\pi/2$, only the even terms in ψ_3 and the odd terms in ψ_4 (cf. 7) will enter. Therefore in ψ_3 only G_0 will change (to G_0') and in ψ_4 only G_{-2} (to G_{-2}'). In order that the wave functions have the proper asymptotic form (only outgoing spherical waves), we must have at large distances

$$\begin{aligned}G_0' &\sim (kr)^{-1}e^{i\eta_0'}\cos(kr+a\log 2kr+\eta_0+\eta_0'),\\ G_{-2}' &\sim (kr)^{-1}e^{i\eta_{-2}'}\cos(kr+a\log 2kr+\eta_{-2}+\eta_{-2}'),\end{aligned} \tag{10}$$

where η_0' and η_{-2}' are the phase shifts introduced by the departure from the pure Coulomb field.

The scattering amplitudes may then be determined in terms of the phase shifts and we find for the asymptotic behavior of the wave functions ψ_3 and ψ_4 the form (4a) with f replaced by $f'=f+\phi_f$ and g replaced by $g'=g+\phi_g$ where the additional amplitudes ϕ_f and ϕ_g are given by

$$\phi_f=\frac{\xi_0}{1+i\xi_0}e^{2i\eta_0},\quad \phi_g=-\frac{\xi_{-2}}{1+i\xi_{-2}}e^{2i\eta_{-2}},\quad \xi_\kappa=-\tan\eta_\kappa'. \tag{11}$$

We may now establish the connection between the phase shifts and the force field by the smooth joining of the radial wave functions inside and outside $r=r_0$. Outside r_0 where the field is Coulombian, the wave function is a linear combination of the regular and irregular solutions for the Coulomb field. We write

$$G_\kappa'=(c_\kappa U_\kappa+\bar{c}_\kappa\bar{U}_\kappa)/kr. \tag{12}$$

The regular and irregular solutions times kr are denoted by U_κ and $\bar{U}_\kappa$, respectively. For the asymptotic behavior we choose U_κ and $\bar{U}_\kappa$ so that at large distances

$$\begin{aligned}U_\kappa&\sim\cos\,(kr+a\log 2kr+\eta_\kappa),\\ \bar{U}_\kappa&\sim\sin\,(kr+a\log 2kr+\eta_\kappa),\end{aligned} \tag{13}$$

and in order to obtain agreement with (10)

$$c_\kappa=e^{i\eta_\kappa'}\cos\eta_\kappa',\quad \bar{c}_\kappa=-e^{i\eta_\kappa'}\sin\eta_\kappa',\quad \bar{c}_\kappa/c_\kappa=\xi_\kappa. \tag{14}$$

We denote kr times the radial functions inside r_0 by u_κ. In the following we shall refer to the functions u_κ simply as the inside wave functions. Aside from the regularity requirement at the origin so that the integrability condition may be fulfilled no restrictions on the inside wave functions need be made. Joining the inside and outside wave functions at r_0 we have

$$\zeta_\kappa\equiv r_0(d\log u_\kappa/dr)_{r_0}=r_0[d\log\,(U_\kappa+\xi_\kappa\bar{U}_\kappa)/dr]_{r_0}. \tag{15}$$

The radial functions U_κ and $\bar{U}_\kappa$ may be easily found in terms of solutions of the differential equation for the confluent hypergeometric function. We find

$$U=(kr)^{-\frac{1}{2}}(\chi+\chi^*),\quad \bar{U}=-i(kr)^{-\frac{1}{2}}(\bar{\chi}-\bar{\chi}^*), \tag{16}$$

where χ and $\bar{\chi}$ are the regular and irregular solutions of

$$d^2\chi/dr^2+(k^2+k(2a-i)/r-(\gamma^2-\tfrac{1}{4})/r^2)\chi=0. \tag{16a}$$

The required solutions are

$$\begin{aligned}\chi&=e^{\pi a/2}\Gamma(\gamma+1+ia)(2^{\frac{3}{2}}\Gamma(2\gamma+1))^{-1}e^{i(\eta_\kappa-\pi/4)}M_{-ia-\frac{1}{2},\,\gamma}(2ikr),\\ \bar{\chi}&=e^{-\pi a/2}2^{-\frac{3}{2}}e^{i(\eta_\kappa+\pi/4)}W_{ia+\frac{1}{2},\,\gamma}(-2ikr),\end{aligned} \tag{17}$$

where M and W are the confluent hypergeometric functions defined e.g., in Whittaker and Watson.[13]

By virtue of the smallness of the range the arguments of χ and $\bar{\chi}$ at $r=r_0$ are very small and the behavior of these functions at the origin may be used. With

$$\left.\begin{aligned}M&\sim(2ikr)^{\gamma+\frac{1}{2}},\\ W&\sim(\Gamma(2\gamma)/\Gamma(\gamma-ia))(-2ikr)^{-\gamma+\frac{1}{2}}\end{aligned}\right\}kr_0\ll 1, \tag{18}$$

we find from (15), (16), (17) and (18)

$$\xi_\kappa=\sigma_\kappa\frac{\gamma-\zeta_\kappa}{\gamma+\zeta_\kappa}, \tag{19}$$

[13] E. T. Whittaker and G. N. Watson, *Modern Analysis* (Cambridge University Press, Third Edition, 1920), Chapter 16.

where $\gamma=(1-\alpha^2Z^2)^{\frac{1}{2}}$ is the same for both values of κ (cf. 9a and 9b), and

$$\sigma_\kappa=\frac{U_\kappa(r_0)}{\bar{U}_\kappa(r_0)}=\frac{e^{\pi a}|\Gamma(\gamma+ia)|^2}{\Gamma(2\gamma)\Gamma(2\gamma+1)}(\gamma\cot\theta_\kappa-a)(2kr_0)^{2\gamma},\quad e^{2i\theta_\kappa}=(\kappa+1+ib)/(\gamma+ia). \tag{20}$$

From the result (19) for the phase shifts and from Eq. (11) for the additional scattering amplitudes the scattering as well as the asymmetry may be determined in terms of the inside wave functions at the boundary $r=r_0$ (i.e., in terms of ζ_κ). Or more suitably the ζ_κ or ξ_κ can be determined by the condition of small asymmetry (or scattering diminished at high energies and unchanged at low energies) and then the scattering (or asymmetry) can be found from these values of ξ_κ. The question of the consistency of small asymmetry and diminished scattering is considered in the next section.

Scattering intensity and asymmetry

We denote the scattering and asymmetry in the perturbed field by I' and $2\delta'^2$, respectively, the unprimed quantities referring to the Coulomb field. Thus the scattering I' is defined (cf. 5) in terms of the elastic scattering cross section Φ', in the perturbed field, by

$$I'=k^2\Phi'=|f'|^2+|g'|^2, \tag{5a}$$

and the asymmetry $2\delta'^2$ is defined by the intensity of the doubly scattered beam which is now proportional to $1+\delta'^2\cos\varphi$. The relative difference between scattering at the azimuthal angles 0 and π is (cf. 6)

$$2\delta'^2=2[i(f'g'^*-f'^*g')]^2/I'^2. \tag{6a}$$

The scattering amplitudes f' and g' are expressed in terms of the corresponding Coulombian amplitudes, f and g, and the additional amplitudes ϕ_f and ϕ_g (cf. remarks preceding 11). It then follows that

$$I'-I=\Delta I=f\phi_f^*+f^*\phi_f+g\phi_g^*+g^*\phi_g+|\phi_f|^2+|\phi_g|^2, \tag{21}$$

$$\delta'I'-\delta I=i(f\phi_g^*-f^*\phi_g)-i(g\phi_f^*-g^*\phi_f)+i(\phi_f\phi_g^*-\phi_f^*\phi_g), \tag{22}$$

where f and g are defined implicitly in the foregoing and are given explicitly by Mott.[1] Inserting ϕ_f and ϕ_g from (11) and denoting the real and imaginary parts of f and g by the subscripts r and i we have

$$\tfrac{1}{2}\Delta I=\frac{\xi_0}{1+\xi_0^2}(\alpha_1+\alpha_2\xi_0)-\frac{\xi_{-2}}{1+\xi_{-2}^2}(\alpha_3+\alpha_4\xi_{-2}), \tag{21a}$$

where the constants α are

$$\begin{aligned}&\alpha_1=f_r\cos 2\eta_0+f_i\sin 2\eta_0, &&\alpha_2=\tfrac{1}{2}+f_r\sin 2\eta_0-f_i\cos 2\eta_0,\\ &\alpha_3=g_r\cos 2\eta_{-2}+g_i\sin 2\eta_{-2}, &&\alpha_4=-\tfrac{1}{2}+g_r\sin 2\eta_{-2}-g_i\cos 2\eta_{-2},\end{aligned} \tag{21b}$$

and

$$\tfrac{1}{2}(I'\delta'-I\delta)=\frac{\xi_0}{1+\xi_0^2}(\beta_1+\beta_2\xi_0)+\frac{\xi_{-2}}{1+\xi_{-2}^2}(\beta_3+\beta_4\xi_{-2}) +\frac{\xi_0\xi_{-2}}{(1+\xi_0^2)(1+\xi_{-2}^2)}[\beta_5(\xi_0-\xi_{-2})+\beta_6(1+\xi_0\xi_{-2})], \tag{22a}$$

where the constants β are

$$\begin{aligned}&\beta_1=g_i\cos 2\eta_0-g_r\sin 2\eta_0, &&\beta_3=f_i\cos 2\eta_{-2}-f_r\sin 2\eta_{-2}, &&\beta_5=-\cos 2(\eta_0-\eta_{-2}),\\ &\beta_2=g_i\sin 2\eta_0+g_r\cos 2\eta_0, &&\beta_4=f_i\sin 2\eta_{-2}+f_r\cos 2\eta_{-2}, &&\beta_6=\sin 2(\eta_0-\eta_{-2}).\end{aligned} \tag{22b}$$

The Eqs. (21a) and (22a) must be solved in terms of a prescribed scattering and asymmetry for ξ_0 and ξ_{-2}. The values of ξ_0 and ξ_{-2} must however be consistent with the relations (19) for ξ_κ in terms of the inside and outside wave functions. From (19) we see that in general $\xi_\kappa \sim \sigma_\kappa$ and because of the small range assumed for the non-Coulombian forces, σ_κ is very small compared to unity for all energies of interest: $(kr_0)^{2\gamma} \leqslant 0.1$ for $E \lessapprox 5.4$ Mev. Therefore both ξ and ξ_{-2} will in general be small of order $\sigma_\kappa \sim 10^{-2}$. While small phase shifts (or phase shifts near a multiple of π) would obviously give Coulombian scattering at small energies, neither the asymmetry (which must be materially reduced at least at small energies) nor the scattering at higher energies would be in accord with the observations.

In order to avoid this difficulty the phase shifts, or better ξ_0 and ξ_{-2}, must be large (of order unity). From (19) it is seen that this requires a logarithmic derivative of the inside wave function at the boundary differing from $-\gamma/r_0$ by terms of order σ_κ; i.e.,

$$\zeta_\kappa = -\gamma + O(\sigma_\kappa). \tag{23}$$

In general this can be the case for only one of the two waves since the two wave functions are not independent. Thus either

$$\zeta_0 = -\gamma + O(\sigma_0), \quad \zeta_{-2} \sim \sigma_{-2} \ll 1, \tag{23a}$$

and only $s_{\frac{1}{2}}$ waves are scattered appreciably, or

$$\zeta_{-2} = -\gamma + O(\sigma_{-2}), \quad \zeta_0 \sim \sigma_0 \ll 1, \tag{23b}$$

and only $p_{\frac{1}{2}}$ waves are scattered.

These conditions on the wave functions at the boundary simply mean that if one of the phase shifts is not small, the irregular part of the corresponding outside wave function is much larger than the regular part at distances small compared to the wave-length. Therefore, in joining the inside and outside wave functions the former is essentially equal to the irregular part of the latter.

We may now investigate the scattering and asymmetry in the two cases: scattering of $s_{\frac{1}{2}}$ waves or of $p_{\frac{1}{2}}$ waves. If, for example, we consider the scattering of $s_{\frac{1}{2}}$ waves we have only one parameter, ξ_0, which may be chosen so as to give the desired asymmetry at a particular energy. Then for a given choice of the non-Coulombian interaction the energy dependence of ξ_0 would be fixed by (19). However, it is more satisfactory to determine the values of ξ_0 which will give the required asymmetry (or scattering) at each energy and then require that the interaction be chosen so as to give the energy dependence for ξ_0 found in this way. Then with the phase shifts determined the scattering (or asymmetry) may be computed.

The numerical results given below will pertain to scattering at 90° in Au since these are the conditions under which most of the double scattering experiments have been carried out. Since the discrepancies observed in the scattering at angles near 90° are fairly typical of most of the angular distribution and since the calculated scattering is not very sensitive at large angles, the conclusions drawn from the results given below will apply to the scattering at other large angles and for elements of neighboring atomic number. For the observed scattering at large energies the data obtained by Barber and Champion[8] for Hg will apply. Thus we must obtain about 15 percent of the Coulombian scattering for energies of the order 500 kev or more and at smaller energies the scattering must be essentially Coulombian.

The constants α_i and β_i occurring in (21a) and (22a) were computed from numerical results for the Coulomb scattering obtained by Bartlett and Watson.[14] The asymmetry, when only one wave is scattered, is given by

$$\tfrac{1}{2}I(\delta'-\delta) = (\xi_0/1+\xi_0^2) \times[\beta_1 - \delta'\alpha_1 + (\beta_2 - \delta'\alpha_2)\xi_0] \tag{22c}$$

for the scattering of $s_{\frac{1}{2}}$ waves and a similar expression when only $p_{\frac{1}{2}}$ waves are scattered. The results for the asymmetry and scattering are given below.

Scattering of $s_{\frac{1}{2}}$ waves.—At low energies, $E \lessapprox 300$ kev, it is impossible to obtain an asymmetry in agreement with experiment. As may be seen from (22c) the asymmetry as a function of the phase shift parameter ξ_0 has a minimum. This minimum asymmetry at various energies is

E(kev)	100	150	300
$200\delta'^2$	5.64	6.56	3.70

[14] I am indebted to Professor Bartlett for making these results available to me before publication.

which is 5 to 6 times larger than the maximum allowable value. At large energies the phase shift parameter was determined so as to give the minimum scattering (cf. 21a with $\xi_{-2}=0$). This minimum scattering at various energies is

E(kev)	500	750	1000	1500
$(I'/I)_{\min}$	0.26	0.29	0.34	0.38

which is somewhat larger than the observed ratios although perhaps acceptable in view of the large experimental error involved in the measurements. The asymmetry which corresponds to this scattering is about the same as the asymmetry in the pure Coulomb field, *viz.* about 5 percent at these energies. While no measurements of the asymmetry have been made at these large energies, it is somewhat implausible that the polarization at large energies should be any greater than that at the lower energies. However, in the absence of such experimental information it seems difficult to exclude this possibility on *a priori* grounds.

Scattering of $p_{\frac{1}{2}}$ waves.—At low energies it is possible to obtain the correct scattering and asymmetry. For an asymmetry of one percent the scattering is

E(kev)	100	150	300
I'/I	0.94	0.92	0.87

which is in satisfactory agreement with the observations. However, at large energies the *minimum* scattering is much too large:

E(kev)	500	750	1000	1500
$(I'/I)_{\min}$	0.81	0.75	0.72	0.69

Thus it is seen that at low energies the scattering of $p_{\frac{1}{2}}$ waves and at high energies the scattering of $s_{\frac{1}{2}}$ waves yields results for the asymmetry and scattering intensity which are roughly in agreement with experiment. Therefore an explanation of the observations in terms of a deviation from the Coulomb field at small distances requires an interaction which will make the inside wave functions fulfill (23b) at low energies and (23a) at high energies. This would imply that the logarithmic derivatives (at the boundary) of the $s_{\frac{1}{2}}$ and $p_{\frac{1}{2}}$ wave functions vary rather rapidly in some intermediate energy range and are practically constant (equal to 0 or $-\gamma/r_0$) at other energies. Unless one assumes an interaction with a rather special type of energy dependence, this behavior of the wave functions is not to be expected. Excluding the possibility of a comparatively abrupt switch from $s_{\frac{1}{2}}$ scattering to $p_{\frac{1}{2}}$ scattering we may conclude that insofar as the experiments are correct, it is impossible to account for the observed results by assuming a small distance interaction which scatters only one of the two waves. This conclusion is, of course, not surprising since with one parameter it is in general rather difficult to account for a scattering which varies as rapidly with energy as the experiments would seem to indicate and at the same time to account for the absence of polarization as is observed in the double scattering experiments.

Scattering of $s_{\frac{1}{2}}$ and $p_{\frac{1}{2}}$ waves.—If the observed scattering and asymmetry are to be accounted for by a non-Coulombian interaction at small distances it seems plausible from the foregoing to require that both waves must be scattered with appreciable phase shifts. In this section we investigate the conditions which must be imposed on the interaction in order that both waves be scattered.

The scattering of both waves implies that for both $s_{\frac{1}{2}}$ and $p_{\frac{1}{2}}$ electrons the condition

$$\zeta_\kappa=-\gamma(1-2\sigma_\kappa/\xi_\kappa(E)+O(\sigma_\kappa^2)) \qquad (24)$$

must be fulfilled. In order that the correct scattering and asymmetry be obtained the values of ξ_0 and ξ_{-2}, regarded as independent variables, must be those obtained from the simultaneous solution of (21a) and (22a) for prescribed scattering and asymmetry. Thus (24) with the proper values of ξ_κ constitutes the necessary and sufficient condition that a small distance interaction account for the scattering and asymmetry. While it is essential that the small term proportional to σ_κ in (24) be accounted for, the first problem to be considered is the conditions under which the logarithmic derivatives ζ_κ/r_0 can be both equal to $-\gamma/r_0$ within small terms of order σ_κ. In addition to (24) it is necessary that the wave functions be regular at the origin. This requirement together with the requirement of an irregular behavior at the boundary implies that in the distance r_0 the wave functions change appreciably. This means a wave-length comparable with r_0 and hence an interaction inside r_0 which is much larger than mc^2.

The condition that the difference $\zeta_0-\zeta_{-2}$ be small can be expressed in the form

$$\zeta_0-\zeta_{-2}=-2\left[\int_0^{r_0} dr[d\log rV/dr)(u_0u_{-2}/rV)\right]\bigg/(u_0u_{-2}/rV)_{r=r_0},\tag{25}$$

$$\sim O(\sigma_\kappa)\ll 1,$$

wherein we have assumed that the small distance interaction V can be represented by an ordinary scalar potential which is the same for both $s_{\frac{1}{2}}$ and $p_{\frac{1}{2}}$ electrons.

A solution of (25) is in general rather difficult. Certain simple cases which can be investigated do not conform to both (25) and the regularity condition at the origin. Thus. (25) may be fulfilled by taking

$$d\log rV/dr=\sigma\rho(r)$$

where ρ is any regular function and of order unity for all $r\leqslant r_0$ while $\sigma\ll 1$. Then V differs from a Coulomb-like potential by terms of order σ. But then the wave-length will be the same as for a pure Coulomb field and the regularity condition cannot be fulfilled. A second example for which the wave functions can be found is the case of a constant potential. For the wave functions which are regular at the origin one finds

$$\zeta_0-\zeta_{-2}=x\cot x-\left(\frac{x^2}{1-x\cot x}-1\right),\qquad x=Vr_0/\hbar c,$$

in which we have used $V\gg mc^2$. Unfortunately there is no value of x for which (25) is satisfied.

It seems that the most plausible solution of (25) is that the interaction V is so large that several oscillations of the wave functions u_0 and u_{-2} may take place in the distance r_0, (large phase shifts!). Then since the phases of the two wave functions will usually be different and since the (variable) wave-lengths will be different, at some points the wave functions will be in phase and at other points out of phase. In this way the integral in (25) may be small by virtue of almost complete interference. The large magnitude of V resulting in many oscillations inside r_0 will allow an irregular behavior at the boundary and a regular behavior at the origin. This then would constitute a necessary though not sufficient condition on the non-Coulombian interaction.