If we multiply $\psi_{lsjm}(\theta_1, \theta_2) = \sum_{m_s m_l} C^{jm}_{lm_l sm_s} \psi_{lm_l}(\theta_1)\psi_{sm_s}(\theta_2)$ by $\psi_{pq}(\theta_1)$.

By using the multiplication theorem, we get:

$$\sum_{l'j'm'} a^{l'j'}_{lj,\,sp} C^{j'm'}_{jmpq} \psi_{l'sj'm'}(\theta_1, \theta_2) = \sum_{l'm_l'm_lm_s} C^{jm}_{lm_lsm_s} b^{l'}_{lp} C^{l'm_l'}_{lm_lpq} \psi_{l'm_l'}(\theta_1)\psi_{sm_s}(\theta_2).$$

If now we multiply by

$$\psi_{l^*sj^*m^*}(\theta_1, \theta_2) = \sum_{m_lm_s} C^{j^*m^*}_{l^*m_lsm_s} \psi_{l^*m_l}(\theta_1)\psi_{sm_s}(\theta_2)$$

we get because of orthogonality

$$a^{l^*j^*}_{lj,\,sp} C^{j^*m^*}_{jmpq} = \sum_{m_lm_s} b^{l^*}_{lp} C^{jm}_{lm_lsm_s} C^{j^*m^*}_{l^*m_l^*sm_s} C^{l^*m_l^*}_{lm_lpq}.$$

By combining the b and a we get

$$B^{l^*j^*}_{lj,\,sp} C^{j^*m^*}_{jmpq} = \sum_{m_lm_s} C^{jm}_{lm_lsm_s} C^{j^*m^*}_{l^*m_l^*sm_s} C^{l^*m_l^*}_{lm_lpq}.$$

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The Static Interaction of Charged Particles

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The energy of the field due to two charges as described by the field equations of the unquantized Born-Infeld theory is calculated to the first approximation in the case of stationary charges not subject to other external fields. The method of representing the field as the superposition of plane waves in a very large cavity is applied to the field equations expanded in a power series in the constant $1/b^2$ of the Born theory. For the first term this leads to the usual Coulomb energy. The process is then extended to calculate the second term from insertion of the coefficients arising in the first term evaluation. In this case it is found necessary to perform threefold averages in place of the previous single averages, since three of the radiation oscillators describing the field enter into combination. The resulting energy is $e_1e_2/r[1-(1/b^2)(e_1^2+e_2^2)/8r^4]$. The nature of the series expansion makes the method invalid in the region near 10^{-13} cm since there the series becomes divergent.

THE derivation of the energy of the field due to point charges from Maxwell's equations suggests that a somewhat similar procedure might be applied to arrive at the corresponding relation for field equations which are slightly different from those of Maxwell. Despite the persistent difficulties arising in the nonclassical, as well as in the classical theories, the belief that some such departure is necessary seems to be well founded. Whether the alteration should be introduced as a semi-classical concept before quantization, or arise in the process of quantization is not yet clear. In any case, it seems probable that for large fields or small distances the classical equations of Maxwell must be modified by the addition of a small nonlinear part. It is the purpose of the present discussion to calculate the interaction

of charged particles on the basis of the Born-Infeld equations in the simplest case, i.e., two stationary charges. The restriction to two particles is imposed only to simplify the index notation and facilitate the integration operations arising in a more complex system. The restriction to stationary charges is a more serious limitation, since in the region of validity of these calculations it is doubtful that any corresponding natural state would be appreciably influenced by the added terms. It seems advisable, however, to view first the simplest consideration, and thus to attempt to prescribe a means of reducing the averaging to a simple set of quadratures. Since the departure must be slight in the classical region (small charges and large inter-particle distances), the energy must closely approximate the Coulomb value in that region. The Coulomb part will, in fact, be taken as the first term of the series representing the total energy.

For the case treated here, Born-Infeld equations reduce to

$$\text{div } \mathbf{D}=4\pi\Sigma e_i\delta(\mathbf{r}-\mathbf{r}_i), \quad \text{curl } \mathbf{E}=0,$$
$$\mathbf{D}=\mathbf{E}(1-E^2/b^2)^{-\frac{1}{2}}, \quad (1)$$

where δ is the Dirac δ function. These would, of course be the usual expressions in free space if $1/b^2$ were zero, hence this quantity will be used as the expansion parameter in the power series. The regions of convergence of such expansions are evidently limited by the relative magnitudes of the field strength and the value of the constant $1/b^2$. Eq. (1) may be written as

$$\text{div } \mathbf{E}(1-E^2/b^2)^{-\frac{1}{2}}=\text{div } (\mathbf{E}+\mathbf{E}E^2/2b^2+\cdots)$$
$$=4\pi\Sigma e_i\delta(\mathbf{r}-\mathbf{r}_i), \quad \mathbf{E}=-\text{grad } \Phi. \quad (2)$$

When the quantities entering into (2), $\mathbf{E}$ and Φ, are expressed as a power series in $1/b^2$ as mentioned, e.g. $\mathbf{E}=\mathbf{E}^{(0)}+\mathbf{E}^{(1)}/b^2+\cdots$, and substituted into (2), one obtains a set of equations by equating coefficients:

$$\text{div } \mathbf{E}^{(0)}=4\pi\Sigma e_i\delta(\mathbf{r}-\mathbf{r}_i), \quad \mathbf{E}^{(0)}=-\text{grad } \Phi^{(0)},$$
$$\text{div } \mathbf{E}^{(1)}=-\text{div } \mathbf{E}^{(0)3}/2, \quad \mathbf{E}^{(1)}=-\text{grad } \Phi^{(1)}, \quad (3)$$
$$\cdots\cdots\cdots\cdots$$

The zero-order term is seen at once to have a solution possessing singularities, whereas $\mathbf{E}$ should remain finite. This inconsistency may be seen in a clearer light by returning for a moment to Eq. (1). The $\mathbf{E}$ value resulting from the distribution of charges e_i may be expressed in terms of $\mathbf{D}$ in a fashion similar to that in the spherically symmetric case. Here $\Sigma e_i/r_i^2$ replaces the single term e/r^2 of the spherically symmetric field. This will show $\mathbf{E}$ to remain finite even at one of the points r_i. However, if one makes use of the expansion of the sort under consideration,

$$\mathbf{E}=\mathbf{D}(1+D^2/b^2)^{-\frac{1}{2}}=\mathbf{D}(1-\tfrac{1}{2}D^2/b^2+\cdots), \quad (4)$$

it is seen that the first term, as well as each of the following, is singular. Actually such an expansion is not convergent for $D^2/b^2\geq 1$, and hence this apparent infinity of the separate terms of $\mathbf{E}$ is not of a troublesome nature.

The solution for the energy from the first-order term follows a procedure somewhat similar to one that has been applied to the equation here designated as the zero order,[1] and the latter will therefore be briefly outlined here in the restricted static case. The calculation cited, being applicable to a quantized non-static case, is more general than the one given below which is changed only in details that make it more adaptable to calculation of the higher order terms in the present consideration.

The Zero-Order Equation

The "hohlraum" method of treating the field as the superposition of plane waves is employed. For convenience, the unit of volume will be chosen so that the space is unity, which space must be very large in comparison to the distance between particles. The frequency density can easily be shown to equal $\nu^2 d\nu/(2\pi c)^3$ per solid angle $d\Omega$. This description of the field gives the zero order potential as

$$\Phi^{(0)}=N\Sigma a_\delta{}^{(0)}\cos(\mathbf{k}_\delta\mathbf{r}), \quad (5)$$

where $N=(8\pi c^2)^{\frac{1}{2}}$ is inserted for ease in handling the numerical factors. Poisson's equation now becomes

$$N\Sigma a_\delta{}^{(0)}k_\delta{}^2\cos(\mathbf{k}_\delta\mathbf{r}_\delta)=4\pi\Sigma e_i\delta(\mathbf{r}-\mathbf{r}_i). \quad (6)$$

If this is multipled by $\cos(\mathbf{k}_\delta\mathbf{r})$ and integrated

[1] E. Fermi, Rev. Mod. Phys. **4**, 87 (1932).

over the whole space,

$$a_\delta^{(0)} k_\delta^2 N = 8\pi \Sigma e_i \cos(\mathbf{k}_\delta \mathbf{r}_i). \quad (7)$$

It may be shown that for this case the Hamiltonian is $\Sigma \frac{1}{2} a_\delta^{(0)2} \nu_\delta^2$. This must be averaged over a finite region of space as well as summed over all directions of $\mathbf{k}_\delta$ and over all values of k (or ν). In the latter case the integration is performed after multiplying by the density function for ν_δ. Upon carrying out these averaging processes for the energy in a finite space, one finds the result to be $\Sigma(e_i e_j)/(2r_{ij})$ where $r_{ij} = r_i - r_j$ (the term involving $r_i + r_j$ has vanished because of its periodicity with respect to the radius vector). Thus, this equation leads to the ordinary Coulomb energy, but is not valid too near the center of charge.

The First-Order Equation

The method considered above will now be extended to the first-order equation of (3). This time three of the radiation oscillators enter into combination to give a pseudo-charge on the right-hand side. Consequently the averages must be taken simultaneously over three of the oscillators, and the integrations as well as the algebraic manipulation become correspondingly more tedious. In the outline of the procedure given below it is found more convenient to designate the negative of a wave number vector by an index different from that applied to the positive direction.

Since $\mathbf{E}^{(0)}$ is defined by $-\text{grad}\ \Phi^{(0)}$, the right-hand side of the equation for $\mathbf{E}^{(1)}$ may be expressed in terms of the $a^{(0)}$ of Eq. (7) This along with the introduction of the new set of ϕ's such that

$$\phi^{(1)} = N \Sigma a_\delta^{(1)} \cos(\mathbf{k}_\delta \mathbf{r})$$

reduces it to

$$\Sigma a_\delta^{(1)} k_\delta^2 \cos(\mathbf{k}_\delta \mathbf{r}) = -(N^2/2) \sum_{\sigma\lambda\mu} a_\sigma^{(0)} a_\lambda^{(0)} a_\mu^{(0)} \times (\mathbf{k}_\sigma \mathbf{k}_\lambda) \mathbf{k}_\mu \cdot (\mathbf{k}_\sigma + \mathbf{k}_\lambda + \mathbf{k}_\mu) \cos[(\mathbf{k}_\sigma + \mathbf{k}_\lambda + \mathbf{k}_\mu)\mathbf{r}]. \quad (8)$$

As before, this is multiplied by $\cos(\mathbf{k}_\delta r)$ and integrated over all space. Because of its periodicity in space, the right-hand side vanishes except for $\mathbf{k}_\delta$ equal to one of the sums $\mathbf{k}_\sigma + \mathbf{k}_\lambda + \mathbf{k}_\mu$. Hence (8) becomes

$$a_\delta^{(1)} k_\delta^2 = -(N^2/2) \Sigma a_\sigma^{(0)} a_\lambda^{(0)} a_\mu^{(0)} (\mathbf{k}_\sigma \mathbf{k}_\lambda)(\mathbf{k}_\mu \mathbf{k}_\delta). \quad (9)$$

The summation is to take place only over values such that $\mathbf{k}_\sigma + \mathbf{k}_\lambda + \mathbf{k}_\mu = \mathbf{k}_\delta$, therefore, in any on the averaging processes the average or summation need be taken only over three of the four k's as the fourth is thereby determined.

It is now necessary to find an expression for the energy in terms of the coefficients "a." Again successive approximations are resorted to and we find

$$4\pi U = b^2[(1 - E^2/b^2)^{-\frac{1}{2}} - 1] = \tfrac{1}{2} E^{(0)2} + 1/b^2 (\mathbf{E}^{(0)} \mathbf{E}^{(1)} + \tfrac{3}{8} E^{(0)4}) + \cdots. \quad (10)$$

When the $\mathbf{E}$'s are replaced and the space averaging performed, it is found that

$$U = \Sigma \tfrac{1}{2} a_\delta^{(0)2} \nu_\delta^2 + 1/b^2 [\Sigma a_\delta^{(0)} a_\delta^{(1)} \nu_\delta^2 + 3\pi c^4 \Sigma a_\sigma^{(0)} a_\lambda^{(0)} a_\mu^{(0)} a_\delta^{(0)} (\mathbf{k}_\sigma \mathbf{k}_\lambda)(\mathbf{k}_\mu \mathbf{k}_\delta)]. \quad (11)$$

The first term the zero-order energy is that already considered in the first part of the discussion. The next two terms, the first-order energy, exhibit a dissymmetry because one of them is written in terms of $a^{(1)}$. By substitution of (9) they may be combined into

$$-\pi c^4 \Sigma a_\sigma^{(0)} a_\lambda^{(0)} a_\mu^{(0)} a_\delta^{(0)} (\mathbf{k}_\sigma \mathbf{k}_\lambda)(\mathbf{k}_\mu \mathbf{k}_\delta). \quad (12)$$

Or by the introduction of the value of the $a^{(0)}$'s they are further reduced to

$$-N^2 \frac{\pi}{C^4} \Sigma e_i e_j e_m e_n \frac{(\mathbf{k}_\sigma \mathbf{k}_\lambda)(\mathbf{k}_\mu \mathbf{k}_\delta)}{(k_\sigma k_\lambda k_\mu k_\delta)^2} \cos(\mathbf{k}_\sigma \mathbf{r}_i) \times \cos(\mathbf{k}_\lambda \mathbf{r}_j) \cos(\mathbf{k}_\mu \mathbf{r}_m) \cos(\mathbf{k}_\delta \mathbf{r}_n). \quad (13)$$

Up to this point the discussion for both the zero and for the first orders in applicable to any number of singularities. The argument will henceforth be restricted to the simplest case, i.e. two charges only. There are three possibilities: (a) two of the subscripts of the r's (or of the e's) are identical, and belong to one of the particles, and the other two are identical, and belong to the second of the particles; (b) three of the subscripts belong to one particle; (c) all of the subscripts belong to one particle.

For case (a), one takes all combinations of equal subscript pairs and performs the space averaging on Eq. (13). The nonvanishing terms must have $\mathbf{k}_\sigma+\mathbf{k}_\lambda+\mathbf{k}_\mu=\mathbf{k}_\delta$, which terms may be shown to vanish[2] upon summing over all directions of $\mathbf{k}$ and integrating over all of its values. The physical meaning of the disappearance of these terms is at once evident if it is noted that the terms would involve $e_1^2e_2^2$ which would appear with the same sign whether the signs of the charges e_1 and e_2 were alike or opposite.

For case (*b*), however, there will be found to be a nonvanishing part. Here let $i=j=m\neq n$, which, upon taking the space average and making use of the fact that $\mathbf{k}_\sigma+\mathbf{k}_\lambda+\mathbf{k}_\mu=\mathbf{k}_\delta$, one finds that (13) reduces to

$$8\pi^3\Sigma e_i^3e_n\frac{(\mathbf{k}_\sigma\mathbf{k}_\lambda)(\mathbf{k}_\mu\mathbf{k}_\delta)}{(k_\sigma k_\lambda k_\mu k_\delta)^2}\cos(\mathbf{k}_\delta\mathbf{r}_{in}). \tag{14}$$

The scalar product part of this may be written as

$$(\mathbf{k}_\sigma\mathbf{k}_\lambda)(k_\delta^2-\mathbf{k}_\delta\mathbf{k}_\sigma-\mathbf{k}_\delta\mathbf{k}_\lambda). \tag{15}$$

For the direction and magnitude integrations of the first term of this, $\mathbf{k}_\sigma$, $\mathbf{k}_\lambda$, $\mathbf{k}_\mu$ are chosen as the variables since k_δ does not occur and it may also be eliminated from $\cos(\mathbf{k}_\delta\mathbf{r}_{in})$ by expanding it in terms of the other $\mathbf{k}$'s. After multiplication by the three density functions, for k_σ, k_λ, k_μ, the first term may now be summed over direction and magnitude for each separately. The final result for this part is, then,

$$-\Sigma e_i^3e_n/8r_{in}^5. \tag{16}$$

In considering the second part of (15) it is more convenient to choose $\mathbf{k}_\lambda$, $\mathbf{k}_\mu$, $\mathbf{k}_\delta$ as the wave number variables. Upon averaging as in the first part, it is found that this part vanishes.

The third part of (15) is exactly the same as the second except that the indices σ and λ are interchanged, hence this part also vanishes.

In case (*c*), one is dealing with the action of the charge on itself and, as pointed out in the first part of the discussion, the arguments used here are not valid when the distances are too small. Case (*c*) may, in fact, be considered as a special case of (*b*) where $i=j=m=n$, and an infinite part is obtained.

Thus for the first-order equation the only term remaining is (16). Combined with the zero order result, the interaction between two stationary particles is therefore given by this calculation to be

$$U=\frac{e_1e_2}{r}\left(1-\frac{1}{b^2}\frac{(e_1^2+e_2^2)}{8r^4}+\cdots\right).$$

The approximation is valid only so long as the value of $1/b^2$ is small enough with respect to the values of e and r that the series converges. In practical considerations, one would generally be concerned with charges in motion, the resulting terms of which may differ considerably from that for the static case.

[2] Except for the self-energy part.