On the Mathematical Formalism of the Theory of Showers

Fundamental equations of the multiplicative theory of showers are so complicated that till now only an approximate treatment has been possible. Bhabha and Heitler give only numerical solutions, Carlson and Oppenheimer introduce an auxiliary equation instead of the primary one. Landau and Rumer in their report to the Moscow nuclear conference, September 1937, emphasizing the non-conclusive character of previous calculations unfortunately arrive at no closed explicit solution of their differential equation, obtained in an elegant manner from fundamental integro-differential equations of the problem. Therefore it seemed to us desirable to seek for a simple closed solution without introducing any new assumptions.

If we choose as the differential cross sections for the pair production by a photon σ_p, and for the radiation of a photon by a charged particle σ_r the expressions

$$\sigma_p = 2K/3E, \quad \sigma_r = K/E, \tag{1}$$

where

$$K = \frac{4NZ^2r_0^2}{137} \ln 183Z^{-\frac{1}{3}},$$

and E is the energy of the photon, we have for variation of the number of particles $P(Et)dE$ and of photons $\Gamma(Et)dE$ as functions of energy E and depth t/K cm

$$\begin{aligned} \frac{dP(Et)}{dt} &= \frac{4}{3}\int_E^\infty d\epsilon \frac{\Gamma(\epsilon t)}{\epsilon} + \int_E^\infty d\epsilon \frac{P(\epsilon t)}{\epsilon - E} - P(Et)\int_0^E \frac{d\epsilon}{E-\epsilon}, \\ \frac{d\Gamma(Et)}{dt} &= \frac{1}{E}\int_E^\infty d\epsilon P(\epsilon t) - \frac{2}{3}\Gamma(Et). \end{aligned} \tag{2}$$

We use now a Laplace-Mellin transformation

$$f(E) = \frac{1}{2\pi i E}\int_{\delta - i\infty}^{\delta + i\infty} ds E^{-s} f(s), \quad f(s) = \int_0^\infty dE E^s f(E)$$

and reduce (2) to differential equations

$$\begin{aligned} \frac{dP(st)}{dt} &= -(\psi(s+1)+\gamma)P(st) + \frac{4}{3(s+1)}\Gamma(st),^1 \\ \frac{d\Gamma(st)}{dt} &= \frac{1}{s}P(st) - \frac{2}{3}\Gamma(st). \end{aligned} \tag{3}$$

With no photons, but a single charged particle of energy E_0 impinging on the boundary, as an initial condition, omitting in (3) the second solution which vanishes for large t, we obtain by means of the saddle point method for evaluation of the integral, the following closed formula for the number of charged particles

$$P(Et) = \frac{\exp\,(\chi(s_0)t)}{E(-2\pi t\chi''(s_0))^{\frac{1}{2}}}\left(1 + 0\left(\frac{1}{t}\right)\right), \tag{4}$$

where

$$\begin{aligned} \chi(s) = \frac{s}{t}\ln\frac{E_0}{E} &- \frac{\psi(s+1)+\gamma+\frac{2}{3}}{2} \\ &+ \left[\left(\frac{\psi(s+1)+\gamma-\frac{2}{3}}{2}\right)^2 + \frac{4}{3s(s+1)}\right]^{\frac{1}{2}} \\ &+ \frac{1}{t}\ln\left(\frac{1}{2} - \frac{\psi(s+1)+\gamma-\frac{2}{3}}{4\left[\left(\frac{\psi(s+1)+\gamma-\frac{2}{3}}{2}\right)^2 + \frac{4}{3s(s+1)}\right]^{\frac{1}{2}}}\right) \end{aligned}$$

and s_0 must be found from the equation

$$\chi'(s_0) = 0.$$

The number of photons is given by an analogous expression.

It is interesting that the conditions of applicability of the saddle point method correspond just to the physical limitation of the whole problem (large t, high energies, neglect of ionization). Our solutions coincide exactly with the results of Carlson-Oppenheimer for $y \gg t$, giving a justification for their rather indirect procedure; but for $y \ll t$ one obtains a slightly more general expression. The use of more general cross sections changes the result only insignificantly.

A more detailed discussion and comparison with the results of other authors will be given elsewhere.

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[1] The definitions of ψ and γ are given in the theory of Γ-functions.

Intense Monochromatic Beams of X-Rays

Dr. Bozorth and Mr. Haworth in their paper on "Focusing of an X-Ray Beam by a Rocksalt Crystal"[1] discuss at the end of their paper a method of obtaining intense monochromatic beams of x-rays which I described in a letter to *Nature*.[2] There are two points in their discussion which I feel need clarification.

The value 2 which they give as the maximum gain obtainable by cutting the crystal as compared to the reflection obtained from a cleavage surface is correct if we only consider the effect of ordinary absorption and secondary extinction. However in monochromation by a single crystal, primary extinction is probably a very important factor. Its influence is difficult to arrive at quantitatively but would, undoubtedly, increase the gain obtainable by this method.

A comparison is made by Bozorth and Haworth of the gains obtainable by the two methods which is very much in favor of the bent crystal method. Such a comparison is not to the point as the monochromatic beams given by the two methods are of a completely different nature. Bent crystal monochromators give a convergent beam. For crystal structure work, such beams are only useful for the study of powders and are not suitable for work with single crystals, fibers or the purpose that I specifically designed the monochromator for, namely the measurement of extremely long spacings.

The possibility of shortening exposures by 50 percent, which has already been realized by the author in practice, is not something to be dismissed lightly when the exposures, even with the modern high power x-ray tubes now available, are of the order of 100 hours.

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[1] Bozorth and Haworth, Phys. Rev. **53**, 538 (1938) (see p. 544).
[2] Fankuchen, Nature **139**, 193 (1937).