The curves derived from this equation are of value for molecular weight determinations. One method[4] of determining molecular weights by the use of the ultracentrifuge depends upon the final equilibrium distribution of concentration having been established. As the solution of the equation shows, this final state is approached asymptotically, it is important to know how long a time must pass before it is approximated sufficiently accurately. The discussion given above of the case of hemoglobin shows that the solution of the equation readily yields this information if an approximate value of the molecular weight can be obtained.

Further, a comparison of the theoretical and experimental distribution curves would be useful for determining whether or not the liquid in the cell was free from stirring. If the liquid was found to be convection free (as in the case of the vacuum type ultracentrifuge[9] and Svedberg's ultracentrifuge[4]), the equation might make it possible to shorten the time of centrifuging by rendering it unnecessary to wait for the final equilibrium state. A knowledge of the sedimentation constant s should allow the molecular weight to be determined from the concentration curve for any time t.

The writer wishes to express his appreciation for the encouragement given him by Professors Beams, Snoddy, McShane and Hoxton in this work.

[9] Beams, J. App. Phys. **8**, 795 (1937).

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Accidental Coincidences in Counter Circuits

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The accidental n-fold coincidences ($n>2$), are of various kinds. The first kind involves n events that accidentally occur nearly simultaneously. The second kind involves two events that are causally simultaneous, and $n-2$ events that are accidentally nearly simultaneous; etc. For theoretical purposes, it is sufficient to treat only accidentals of the first kind, since the theory of the other kinds is very similar. It is also possible, for theoretical purposes, to replace each of the various coincidence circuits in actual use by the same schematic circuit. This consists of three parts: the counting, relay, and recording circuits. For n-fold coincidences, the relay circuit will contain n relays (e.g. type 57 tubes). It is shown that the time constants (τ_i) of these relays determine the accidental counting rates. The time constants of the counting and recording circuits are not important in this connection. It is shown experimentally that each of the relays ordinarily has a different time constant, τ_i, even though it is constructed of commercially identical parts. By the use of these results, it is shown that when N_i particles per second traverse the ith counter, the rate at which accidentals of the first kind occur is given by

$$\begin{aligned} A_{1,\,2,\,\cdots\,n} &= BC, \\ B &= N_1\tau_1\cdot N_2\tau_2\cdots N_n\tau_n, \\ C &= 1/\tau_1+1/\tau_2+\cdots+1/\tau_n. \end{aligned}$$

The rates of occurrence of the other kinds of accidentals are readily deduced from this result by slight modifications. This formula is verified by experiment.

1. Analysis of the Problem, and Schematization of the Circuit

The increasing precision to which counter measurements are being carried makes it necessary to consider the statistical theory of accidental coincidences in greater detail than has hitherto been done. When n counters are connected in such a way that only n-fold coincidences are recorded, the recording rate is the sum of various contributions. Every coincidence consists of n events that occur simultaneously, or nearly so. If all n events are causally related, they form a true coincidence. If no two are causally related, they form an accidental of the first kind. If two are causally related to each other, but no other causal relations exist among the n events, they form an accidental of the second kind. The terminology may be extended because there is one kind of accidental for every partition of the number n into a sum of integers.

Fortunately, the theories of the various kinds of accidentals are very similar, so that it is sufficient if the theory of the first kind is considered in detail. The theories of the other kinds require only that trivial complications be introduced. Consequently, the unmodified term "accidental" will be used to refer to accidentals of the first kind.

There is general agreement on the formula for twofold accidentals: if the rates of the two counters are N_1 and N_2 and the resolving time is τ, the rate at which accidentals occur is

$$A_{12}=2N_1N_2\tau. \tag{1}$$

For threefold accidentals, the rate is

$$A_{123}=\alpha N_1N_2N_3\tau^2, \tag{2}$$

but various values of α have been used. The correct value is $\alpha=3$, as was shown by Schiff,[1] but $\alpha=4$ and $\alpha=6$ have been used. Not much work has required higher values of n.

The validity of these equations depends on assumptions concerning the action of the counters and circuit. Although there are several circuits in use, it appears that all can be schematically represented[2] by Fig. 1. When one of the counters, C, is actuated, it furnishes a pulse of current to the corresponding relay. The relay thereupon closes and the duration of the contact is τ. The recorder is actuated only if all relays make contact simultaneously.

There is some question as to what will occur if the same counter is actuated twice within a time of length τ. It is also likely that different actual circuits will behave differently, and that σ, the duration of the pulse of current delivered by the counter, will be important in this connection. Schiff (p. 94) apparently assumes that $\sigma \ll \tau$ in order to avoid this question. This appears to be a very dubious assumption; in the following derivation, the question will be evaded by supposing that the counting rates are so low that the probability that one counter is actuated twice within a time τ is negligible. This makes the efficiency of the relay circuit 100 percent.

The duration of the pulse of current delivered by the relay circuit to the recorder will not be τ, but will vary from 0 to τ, depending on the timing of the events involved in the accidental. (For true coincidences, this duration will always be τ, unless there should be delays between the occurrence of the events in the counter and the closing of the relays.) The simplest assumption to make concerning the action of the recorder, is that all those pulses delivered by the relay, whose duration is greater than ρ, will be recorded. Pulses whose duration is less than ρ are not recorded. This limit, ρ, must be distinguished from the resolving time ρ' of the recorder: if the relay delivers two pulses whose separation in time is less than ρ', only one of these will be recorded. The influence of ρ' will again be ignored by assuming a sufficiently low counting rate. It will be seen that the influence of ρ is the same as if τ were diminished to $\tau-\rho$.

It is doubtful if any actual circuit corresponds exactly to the scheme of Fig. 1. The experimental investigation described below indicates that the idealization is not too great, provided only that each relay be supposed to have a different contact time, τ_i. This is found to be the case even when commercially identical parts are used in the construction of the relays.

2. Calculation of the Accidental Rates

Let the relays be numbered $i=1, 2, \cdots n$, and let N_i be the independent rate of relay i. That is, N_i is the probable number of events per second that close relay i, *and are not causally related to some event that closes some other relay.*

A coincidence is an interval of time during which all relays are closed. Such a concatenation of events is diagrammed in Fig. 2. The horizontal arrows, τ_i, extend from the instant at which relay i closes until the instant at which it opens again. The lengths, τ_i, of the arrows are thus the contact times of the various relays. The coinci-

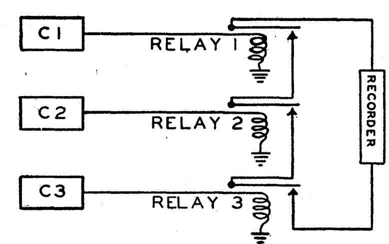

FIG. 1. Schematic circuit for recording coincidences.

[1] L. I. Schiff, Phys. Rev. **50**, 88 (1936).

[2] An alternative schematization would consist of relays in parallel, the recorder being actuated only when all relays are open simultaneously. The theory of such a circuit is identical with that of Fig. 1.

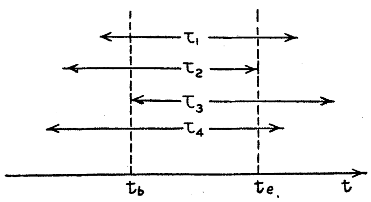

FIG. 2. Diagram illustrating a fourfold coincidence.

dence begins at t_b, the instant at which the last relay (No. 3 in the figure) closes, and ends at t_e, the instant at which the first relay (No. 2 in the figure) opens.

The problem is to calculate $P(t, t')dt$, the probability that n independent events fall together in such a way as to constitute a coincidence with $t<t_b<t+dt$ and $t'<t_e$. As a preliminary, it is convenient to calculate $Q(t, t')$, the probability that the accidental coincidence occurs so that $t_b<t$ and $t'<t_e$. The relation between P and Q is then clearly

$$P(t, t')=\partial Q(t, t')/\partial t. \qquad (3)$$

If it is permissible to ignore the probability that the same relay is actuated twice during its contact time, then $Q(t, t')=0$ unless $t'-t>\tau_i$ for all relays i. The probability that relay i will close during the interval $t'-\tau_i \rightarrow t$ is $N_i(\tau_i-t'+t)$, so that

$$Q(t, t')=\Pi_{i=1}{}^n N_i(\tau_i-t'+t). \qquad (4)$$

Because of the assumption concerning the action of the recorder, the rate at which accidentals are recorded will be $P(t, t+\rho)$. From (4) it is obvious that this is the same as $P(t, t)$, provided that τ_i is replaced by $\tau_i-\rho$. Hence, if the τ_i are determined empirically from the recorded counts, it is permissible to set $\rho=0$. Therefore the accidental rate of n-fold coincidences is

$$\begin{aligned} A_{12\cdots n}&=P(t, t)\\ &=BC, \end{aligned} \qquad (5)$$

where

$$\begin{aligned} B&=N_1\tau_1\cdot N_2\tau_2\cdots N_n\tau_n,\\ C&=1/\tau_1+1/\tau_2+\cdots+1/\tau_n. \end{aligned} \qquad (6)$$

For twofold circuits, this reduces to

$$A_{12}=N_1N_2(\tau_1+\tau_2), \qquad (1')$$

and for threefold, to

$$A_{123}=N_1N_2N_3(\tau_1\tau_2+\tau_2\tau_3+\tau_3\tau_1). \qquad (2')$$

If all the τ_i are equal, these reduce to (1) and (2), with $\alpha=3$.

As has been said, these formulae apply only to accidentals of the first kind. When threefold coincidences are recorded, accidentals of the second kind will also occur. The problem is then to calculate the total rate, N, at which coincidences are recorded, given that

(a) the independent rate of relay i is N_i;
(b) on the average, relays i and j are actuated N_{ij} times per second by two events that are causally related to each other, but are not causally related to an event that actuates counter k;
(c) on the average, true triple coincidences occur N_{123} times per second.

Let τ_{ij} be the smaller of τ_i and τ_j; then the probable number of accidentals of the second kind arising from causally related events in counters 1 and 2, and an unrelated event in counter 3 is easily seen to be

$$A_{12;\,3}=N_{12}N_3(\tau_{12}+\tau_3). \qquad (7)$$

The total rate of recording will thus be

$$N=N_{123}+A_{12;\,3}+A_{23;\,1}+A_{31;\,2}+A_{123}. \qquad (8)$$

If all the τ_i are equal, this reduces essentially to Schiff's Eq. (14).

3. Discussion of Actual Circuits

In order to show that the foregoing assumptions regarding the time constants apply to all circuits in use, the various types of circuits will be analyzed. It is most convenient to consider all these circuits (such as the Neher-Harper, the direct resistance coupled, and others) as composed of three separate circuits: the counting circuit, the relay circuit, and the recording circuit.

A typical onefold circuit is shown in Fig. 3. It consists of a Geiger-Müller tube, an applied voltage on the tube, and some arrangement for

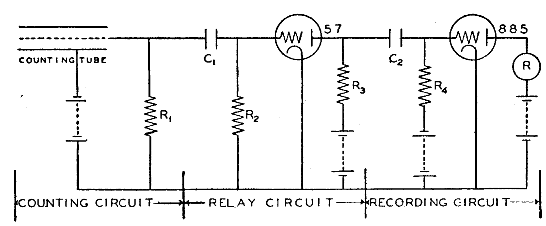

FIG. 3. Single-fold counting circuit.

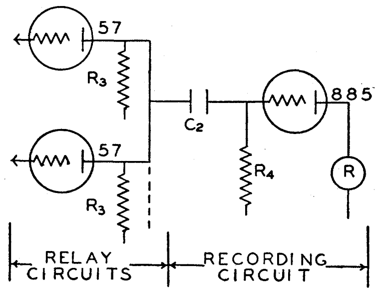

FIG. 4a. Older relay and recording circuit for coincidences.

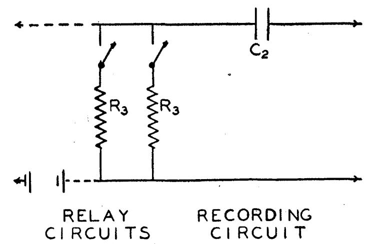

FIG. 4b. Schematization of the same.

quenching the glow in the tube when a discharge occurs. With an activation of the Geiger-Müller tube, a surge of current is produced in the counting circuit, resulting in the delivery of a pulse of duration σ to the relay circuit. This is true regardless of the type of circuit used for the quenching of this glow. σ depends on the characteristics and operating conditions of the Geiger-Müller tube, and the constants of the remainder of the counting circuit. σ is probably not only dependent on the constants of the circuit, and the characteristics of the Geiger-Müller tube, but also on the voltages, the counting rate of the tube, etc.

The relay circuit, consisting of an amplifier, acts as a switching device which sets off the recording circuit. The plate current is normally constant but is reduced nearly to zero by the pulse from the counting circuit. The effective time during which the current is zero is called τ. It is easily seen that the duration of the pulse in the relay circuit must be at least as great as the effective duration of the pulse in the counting circuit. (See Part 1.) Since the value of τ depends on the value of σ, τ will likewise be affected by the factors mentioned above.[3]

The recording circuit is controlled by the relay circuit. The pulse produced in the latter trips the recording mechanism, which in this case consists of a thyratron tube and a mechanical count recorder, R.

Two types of relay circuits have been in use for obtaining coincidence records. The first type is shown in Fig. 4a where each type 57 tube corresponds to one of the relays of Fig. 1. The pulse in the recording circuit is due to the individual pulses in the relay circuits and is equal to their sum. This may be represented schematically by an arrangement of switches and resistances, as shown in Fig. 4b, where the count recorder is so adjusted that it registers only the pulses occurring when all relay switches are closed simultaneously. Pulses of lesser magnitude do not affect the count recorder. This is in agreement with the schematization used above.

The type of circuit shown in Fig. 4a has been superceded by the one shown in Fig. 5a. In this arrangement the current in resistance R_3 is the sum of the currents in all the relays. If the current in some of the relays is reduced to zero, that in R_3 is not appreciably changed[4] because the remaining relays supply more current than before. This is due to the fact that there is a slight increase of voltage on the plates of the remaining type 57 amplifying tubes and the fact that the effective resistance of the tubes is not constant. This type of coincidence circuit is shown schematically in Fig. 5b. The counting mechanism is now so arranged that a count will be registered only when the current in R_3 is greatly reduced. Hence a coincidence will occur only when in an interval of time all the relay switches are opened. This too is in agreement with the schematic assumption made above.

4. EXPERIMENTAL VERIFICATION

The apparatus used in the experimental verification of formula (5) consisted of a threefold Geiger-Müller tube counting system as shown in

[3] In the course of the experiment τ was found to vary somewhat when the individual counting rate was varied. The dependence of τ on such factors is being studied at the present time.

[4] The following experiment was performed with a fourfold counting system: the current in R_3 was measured with a milliammeter, and the currents in the relays were successively reduced to zero by putting a negative bias on the control grids of the type 57 tubes. It was found that the current in R_3 did not vary by more than 3 percent, until all relays were biased, when it dropped to zero.

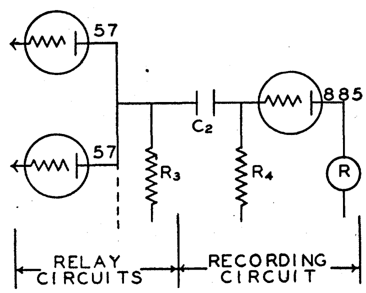

FIG. 5a. Newer relay and recording circuit for coincidences.

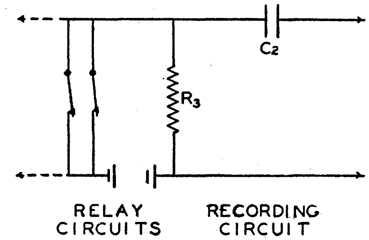

FIG. 5b. Schematization of the same.

Fig. 3 and Fig. 5a. Each of the three branches of this counting system was heavily shielded from electrostatic disturbances.

The Geiger-Müller tubes were made by a special technique developed in this laboratory.[5] They have a threshold voltage of about 1100, and a plateau width of about 800 volts and are filled with hydrogen. Their metal cylinder is 9/16 inch in diameter and $15\frac{1}{2}$ inches in length. Their efficiency is high, and the single counting rate, produced by cosmic rays and local radiation is about 150 per minute when the tubes are not shielded, and is decreased to about 60 per minute when they are heavily shielded with lead.

To verify the formula for accidental counts, it was necessary to have an experimental determination of A_{123}. To obtain this with the ordinary circuit constants would be difficult, since with the ordinary value of τ only one accidental in a few hours would be expected.

Furthermore, the cosmic rays would make the terms N_{123}, $A_{12;\,3}$, etc., in (8) much greater than A_{123}. It was therefore necessary to increase A_{123} by increasing the τ_i and N_i. This was done by using abnormally large resistances and capacities in the counting and relay circuits, and by artificial activation from radioactive sources. The quantities N_{123}, N_{12}, etc., were also decreased by shielding two of the Geiger-Müller tubes with 2.5 inches of lead, and by placing all three out of line in a horizontal plane, at distances of 12 feet from each other. In this way, the total number of recorded counts was made equal to A_{123}.

TABLE I. *Typical set of data. The A's and N's are in counts/min., the τ's in minutes.*

$N_1=182\pm1$	$A_{12}=16.1\pm0.2$	$\tau_1=12.0\pm0.3\ \times10^{-5}$
$N_2=435\pm2$	$A_{23}=21.0\pm0.2$	$\tau_2=\ 8.3\pm0.2\ \times10^{-5}$
$N_3=481\pm2$	$A_{31}=12.0\pm0.1$	$\tau_3=\ 1.7\pm0.02\times10^{-5}$

By means of suitable switches in the circuits, the quantities N_1, N_2, N_3, A_{12}, A_{23}, and A_{31} were also measured. From these and (1′), values of τ_1, τ_2, and τ_3 were calculated. These three numbers were not equal; a typical set of data is shown in Table I.

These data were obtained with a relay circuit whose homologous parts consisted of commercially identical resistors, condensers, and type 57 tubes. When these parts were interchanged, the values of τ_i altered. Thus the practical importance of the generalization of (1′) is demonstrated.

From the values of τ_i obtained above, one calculates $A_{123}=0.51\pm0.03$ min^{-1} from (2′). The value of A_{123} observed under these conditions was 0.513 ± 0.007. Several series of data similar to this were obtained; in no case was the difference between the calculated and observed values of A_{123} greater than 6 percent. Eq. (2′) was also checked by V. C. Wilson for the apparatus described by him in a recent paper.[6]

[5] J. B. Hoag, *Electron and Nuclear Physics* (D. Van Nostrand Company, 1938), p. 432.

[6] V. C. Wilson, Phys. Rev. **53**, 337 (1938).