the Na there was present perhaps as much as 10^{-5} mm pressure of H_2. The possibility of some contamination by H_2 is indicated by the extreme photosensitivity of the Na coated cathode. Such activity in photo-cells is indicative of hydride formation on the Na surface. In any case, it would not be difficult to ascribe a very material destruction of the N_2 excited and metastable states which could have been active at the cathode in the absence of Na to the presence of H_2 contamination.

When the effects of the peaks are deducted, all of the curves indicate a rise from zero that is fairly sharp and in the X/p region between 200 and 300. In this region the positive ions strike the cathode with energies in the range of two to five volts. It is natural to ascribe these rises to positive ion bombardment of the cathodes. These rises, then, represent the true γ proposed by Townsend and Thomson and strongly favored by Holst, Oosterhuis, Penning and Druyvesteyn.

In conclusion the writer desires to express his thanks to Professor L. B. Loeb under whose direction this work was done, for his guidance in its prosecution and for his aid in the interpretation of the results. His thanks are also due to Mr. W. R. Stamper for his aid in the construction of the apparatus.

FEBRUARY 15, 1938 PHYSICAL REVIEW VOLUME 53

Magnetic Interaction in Homogeneously Strained Ferromagnetic Crystals

L. W. McKeehan
Sloane Physics Laboratory, Yale University, New Haven, Connecticut
(Received January 3, 1938)

A well-known method of computing magnetostrains, as the strains necessary to minimize the sum of magnetic and elastic potential energies, is improved in detail, extended to include terms dependent upon the structure of atomic magnets, and is applied in the cases of body-centered cubic iron, face-centered cubic nickel and hexagonal close-packed cobalt. It is shown that magnetostriction is not important in fixing directions of easiest magnetization in these crystals.

IN a crystal composed of atoms having permanent magnetic moments the potential energy density includes terms depending upon these moments and upon their mean relative positions. Such magnetic terms are therefore strain-sensitive. The potential energy density also contains terms depending only upon the strain components of any existing strain and upon the several elastic moduli characteristic of the crystal. It is apparent, therefore, that a crystal under no applied forces will assume whatever state of strain and magnetization that may be required to make the sum of its magnetic and elastic potential energies minimum. If both strain and magnetization are uniform throughout the crystal the total potential energy is proportional to the potential energy density, also homogeneous, which will be denoted by E.

In nonferromagnetic crystals the magnetization under no forces will be zero and the self-imposed strain associated with the presence of atomic magnets cannot be separated from strains associated with other interatomic forces. In a ferromagnetic crystal the magnetization persists under no applied magnetic forces, so that the equilibrium between magnetic and elastic forces can be investigated. The magnetic part of the energy E, say E_m, will in general depend upon the direction along which the axes of the atomic magnets are lined up, that is upon the unit vector $\mathbf{p}$ in the direction of magnetization. In one case of great interest E_m seems at first glance to be independent of $\mathbf{p}$.

If the atomic magnets are magnetic dipoles (point magnets) and if their loci have strictly cubic symmetry, E_m in an unbounded crystal is independent of $\mathbf{p}$. As Akulov[1] first pointed out, this equilibrium at zero strain is unstable. If the elastic potential energy is denoted by E_e the

[1] N. S. Akulov, Zeits. f. Physik **52**, 389–405 (1928).

state of affairs gives $E_m=0$ and $E_e=0$ so that $E_m+E_e=0$. But E_m is a linear function of strain components, E_e is a quadratic function, so that a type of strain can always be found for which E_m begins to grow negatively faster than E_e grows positively, so that $E_m+E_e<0$ for small strains of this type. The same difference in the functions E_m and E_e ensures that a limiting strain will be reached, so that E_m+E_e will always have a minimum. At this minimum it is easy to see that $E_m=-2E_e$. It is also clear that the necessary strain (magnetostrain) will not be a mere dilatation.

In the absence of an applied magnetic field, $\mathbf{p}$ will automatically find one of the so-called directions of easiest magnetization, for which the minimum in (E_m+E_e) is lowest. In a magnetic field $\mathbf{H}_a$ not along one of the directions of easiest magnetization the potential energy density contains the additional term $-(\mathbf{H}_a\cdot\mathbf{I})=-(\mathbf{H}_a\cdot\mathbf{p})I$ and a different state of strain will be necessary to minimize

$$E=E_m+E_e-(\mathbf{H}_a\cdot\mathbf{p})I. \tag{1}$$

For random directions of $\mathbf{H}_a$ the problem is complicated by the difficulty of finding the equilibrium direction for $\mathbf{p}$, which is explicitly involved in the last term on the right-hand side of Eq. (1), implicitly in E_m. If $\mathbf{H}_a$ is along any axis of symmetry in the crystal it is possible to find a magnitude for H_a high enough to keep $\mathbf{p}$ parallel to $\mathbf{H}_a$. Under these conditions it is only necessary to minimize (E_m+E_e) with $\mathbf{p}$ constant.

In noncubic dipole arrays and in all cases where atomic magnets are more extensive than dipoles E_m is no longer zero at zero strain for all directions of $\mathbf{p}$. Denote its more general value at zero strain by ${}_0E_m$.

In an earlier paper,[2] hereafter designated as l.c., formulae for ${}_0E_m$, there called E, are given for cubic and hexagonal crystals with atomic magnets more complicated than dipoles, and distributions of magnetic moment making differences in ${}_0E_m$ fairly consistent with observed differences in (E_m+E_e) for various important directions of $\mathbf{p}$ in iron, nickel, cobalt and some of their alloys, are suggested. In this analysis the quantities compared actually differed by $(E_m-{}_0E_m+E_e)=-E_e$, and it was pointed out in l.c.[2] that including this difference might somewhat alter the results there presented. The present analysis allows a direct computation of the magnetostrain as above defined and shows that the error committed in ignoring $-E_e$ was actually negligible.

The method here used for computing E_m is based upon that of Akulov,[1] and is straightforward. We start with Eq. (6) of l.c.,[2] omit sextupole terms to shorten subsequent work, and have

$${}_0E_m=-(NP^2/Va^3)[F_2S_2+F_4S_4]. \tag{2}$$

Then in general

$$E_m=(1+\mathbf{A})\,{}_0E_m, \tag{3}$$

where $\mathbf{A}$ is an operator equivalent to the application of a homogeneous strain of tensor components, A_{ii}, A_{ij}. The convention regarding subscripts is that i, j, k always refer to different rectangular coordinate axes, as many of which as possible are chosen parallel to crystallographic axes. This necessitates writing both A_{ii} and A_{ij} for the unrestricted tensor component. Furthermore, each permutation of subscripts is counted separately, so that, for example, $\sum A_{ij}$ contains six terms rather than three.

In the notation of Eqs. (2) and (3) what Akulov really did was to compute an energy density

$${}_VE_m=-(NP^2/V)(1+\mathbf{A})(a^{-3}F_2S_2). \tag{4}$$

This amounts to supposing that the volume, V, of a unit cell of the space-lattice remains unaltered no matter how its dimensions are changed by straining. In first recomputation of the coefficients appearing in E_m this procedure was followed,[3] but now the more logical procedure of Eq. (3), in which V may vary, is preferred.

Both S_2 and S_4 are put into rapidly convergent forms before operator $\mathbf{A}$ is applied. The magnetic moment per atom, P, and its distribution about the atom center, which fixes F_4, are regarded as invariant under $\mathbf{A}$. The integers N and $F_2(=1)$ are naturally also invariant. The assumption that P is unaffected by strain re-

[2] L. W. McKeehan, Phys. Rev. **52**, 18–30, 527 (1937). It may be noted, though unimportant here, that Eq. (29a) really gives $(21/20)S_{60}$, Eq. (29b) gives $(99/10)S_{60}$, not S_{60} in each case as printed.

[3] L. W. McKeehan, Phys. Rev. **43**, 1022–1024 (1933).

TABLE I. (Part 1) *Formula for the coefficients of* S_{2+}.

$\gamma_1 = \epsilon^3 \Sigma n_r g_1(x)$
$\gamma_2 = \epsilon^3 \Sigma n_r x^2 g_2(x)$
$\gamma_3 = \epsilon^3 \Sigma n_r x^4 g_3(x)$
$\gamma_4 = \epsilon^3 \Sigma n_r [r_i^2 r_j^2]^* r^{-4} x^4 g_3(x)$
$\gamma_5 = \epsilon^3/3(\pi)^{\frac{1}{2}}$
$\gamma_6 = \pi \Sigma f n_q \exp(-\pi^2 q^2/\epsilon^2)$
$\gamma_7 = \pi \Sigma f n_q (1+\pi^2 q^2/\epsilon^2) \exp(-\pi^2 q^2/\epsilon^2)$
$\gamma_8 = \pi \Sigma f n_q [q_i^2 q_j^2] q^{-4} (1+\pi^2 q^2/\epsilon^2) \exp(-\pi^2 q^2/\epsilon^2)$

Coefficients in Eq. (6)

	γ_1	γ_2	γ_3	γ_4	γ_5	γ_6	γ_7	γ_8
S_{200}	−1/2	+1/6			+2	−2/3		
S_{2ii}	+1/2			−1/2	−2	+4/3		−4
S_{21111}		+1/3	−1/6	+3/2		+4/3	−4/3	+12
S_{22323}		+1/3		−1		+4/3		−8

* Square brackets are used to indicate mean values of the quantities enclosed.

quires further justification. If this be admitted, however, the constancy of F_4 is highly probable.

Equation (3) can be expanded like Eq. (2) so as to separate terms arising in S_2 from those arising in S_4. Thus

$$E_m = -(NP^2/Va^3)[F_2 S_{2A} + F_4 S_{4A}] \quad (5)$$

where the terms in square brackets are pure numbers, F_2 and F_4 being the same as in Eq. (2). Also as before S_{2A} and S_{4A} can be separated into terms depending upon individual strain components and functions of p_1, p_2, p_3 the components of **p**, that is the direction cosines of the magnetization. The results are, for cubic crystals,

$$S_{2A} = S_{200} + S_{2ii}\sum A_{ii} + S_{21111}\sum A_{ii} p_i^2 + S_{22323}\sum A_{ij} p_i p_j, \quad (6)$$

TABLE II. *Numerical values of* S_{2+} *and* S_{4+} *for cubic crystals.*

	PRIMITIVE	BODY-CENTERED	FACE-CENTERED
S_{200}	2.0944	4.1888	8.3776
S_{2ii}	−1.0162	−8.1417	−15.9639
S_{21111}	9.5177	−0.7077	−2.3736
S_{22323}	2.1564	−7.9058	−15.1727
S_{400}	3.1082	−3.1065	−7.5257
S_{4ii}	−6.7567	12.9293	0.3118
S_{41111}	27.5736	83.6175	−355.6216
S_{42323}	0.9471	23.7591	−66.4842
$S_{4111111}$	−32.1692	−97.5538	414.8918
$S_{4ii2233}$	9.9984	−53.2277	88.1260
$S_{4232311}$	−6.6296	−166.3138	465.3894

$$S_{4A} = S_{400}\{1-(5/2)\sum p_i^2 p_j^2\} + S_{4ii}\sum A_{ii} + S_{41111}\sum A_{ii} p_i^2 + S_{42323}\sum A_{ij} p_i p_j + S_{4111111}\sum A_{ii} p_i^4 + S_{4ii2233}(\sum A_{ii})\sum p_i^2 p_j^2 + S_{4232311}\sum A_{ij} p_i p_j p_k^2. \quad (7)$$

Subscripts to S are assigned by adding to 2 or 4 all the subscripts appearing in the unabbreviated first term in the affected sum when arranged in natural cyclic order. Letters *ii* appear if $(\sum A_{ii})$ is a factor, and terms of this sort have been picked out wherever possible so that the effect of dilatationless strains will be easier to calculate.

All the coefficients S_{2+} and S_{4+} are numbers to be found by suitable summations over the points at which atomic magnets are centered, which usually include points of the space-lattice defining the crystal, and over the points of a lattice reciprocal to that of the crystal. Some of these have been given in l.c.[2] and in previous papers

TABLE I. (Part 2) *Formulas for the coefficients of* S_{4+}.

$\delta_1 = \epsilon^5 \Sigma n_r g_2(x)$
$\delta_2 = \epsilon^5 \Sigma n_r x^2 g_3(x)$
$\delta_3 = \epsilon^5 \Sigma n_r x^4 g_4(x)$
$\delta_4 = \epsilon^5 \Sigma n_r [r_i^2 r_j^2] r^{-4} x^4 g_4(x)$
$\delta_5 = \epsilon^5 \Sigma n_r x^6 g_5(x)$
$\delta_6 = \Sigma^5 \Sigma n_r [r_i^2 r_j^2] r^{-4} x^6 g_5(x)$
$\delta_7 = \epsilon^5 \Sigma n_r r_1^2 r_2^2 r_3^2 r^{-6} x^6 g_5(x)$
$\delta_8 = \epsilon^5/5(\pi)^{\frac{1}{2}}$
$\delta_9 = \pi^3 \Sigma f n_q q^2 \exp(-\pi^2 q^2/\epsilon^2)$
$\delta_{10} = \pi^3 \Sigma f n_q [q_i^2 q_j^2] q^{-2} \exp(-\pi^2 q^2/\epsilon^2)$
$\delta_{11} = \pi^3 \Sigma f n_q q^2 (1+\pi^2 q^2/\epsilon^2) \exp(-\pi^2 q^2/\epsilon^2)$
$\delta_{12} = \pi^3 \Sigma f n_q [q_i^2 q_j^2] q^{-2} (1+\pi^2 q^2/\epsilon^2) \exp(-\pi^2 q^2/\epsilon^2)$
$\delta_{13} = \pi^3 \Sigma f n_q q_1^2 q_2^2 q_3^2 q^{-4} (1+\pi^2 q^2/\epsilon^2) \exp(-\pi^2 q^2/\epsilon^2)$

Coefficients in Eq. (7)

	δ_1	δ_2	δ_3	δ_4	δ_5	δ_6	δ_7	δ_8	δ_9	δ_{10}	δ_{11}	δ_{12}	δ_{13}
S_{400}	+1/8	−1/12	+1/72	−1/12				−1	+2/9	−4/3			
S_{400}			−1/180	+1/12					−4/45	+4/3			
S_{4ii}	−1/8	+1/24	−1/72	+1/3		−1/48	+1/48	+1	−4/9	+8/3		+2/3	−2/3
S_{41111}		−1/6	+1/12	−1/4		−1/8	+3/8			−8		+4	−12
S_{42323}		−1/6	+1/36	+7/12		−1/12	+1/12		−4/9	−4/3		+8/3	−8/3
$S_{4111111}$			+1/18	−5/6	−1/72	+13/48	−7/16		−8/9	+40/3	+4/9	−26/3	+14
$S_{4ii2233}$			+1/72	−5/24		+1/48	−7/48		+4/9	−20/3		−2/3	+14/3
$S_{4232311}$			−1/36	+5/12		+1/12	−7/12		+4/9	−20/3		−8/3	+56/3

of this series, but formulae for all of them are presented in Table I, Parts 1 and 2 which assigns an abbreviating symbol to each different summation needed and expresses each S in terms of these symbols.

Numerical values of each S are given in Table II for the three cubic arrangements. All values given have been checked to or beyond the last digit here recorded by computations with two different values of ϵ. (Values of S_{200} have also been checked by simpler formulae.)

For hexagonal crystals Eq. (5) splits up differently and yields the following equations:

$$\begin{aligned}S_{2A}&=S_{200}+S_{20033}p_3^2+S_{2ii}\textstyle\sum A_{ii}+S_{233}A_{33}\\&+S_{21111}(A_{11}p_1^2+A_{22}p_2^2)+S_{2ii33}(\textstyle\sum A_{ii})p_3^2\\&+S_{23333}A_{33}p_3^2+S_{21212}(A_{12}+A_{21})p_1p_2\\&+S_{21313}(A_{13}p_1+A_{23}p_2)p_3\\&+S_{23113}(A_{31}p_1+A_{32}p_2)p_3,\end{aligned}\tag{8}$$

$$\begin{aligned}S_{4A}&=\{S_{400}+S_{4ii}\textstyle\sum A_{ii}+S_{433}A_{33}\}\\&\times\{1-10p_3^2+(35/3)p_3^4\}+S_{41111}A_{11}p_1^2\\&+S_{41122}A_{11}p_2^2+S_{41133}A_{11}p_3^2+S_{42211}A_{22}p_1^2\\&+S_{42222}A_{22}p_2^2+S_{42233}A_{22}p_3^2\\&+S_{41212}(A_{12}+A_{21})p_1p_2+S_{41313}(A_{13}p_1+A_{23}p_2)p_3\\&+S_{43113}(A_{31}p_1+A_{32}p_2)p_3+S_{4111111}A_{11}p_1^4\\&+S_{4112222}A_{11}p_2^4+S_{4113333}A_{11}p_3^4\\&+S_{4221111}A_{22}p_1^4+S_{4222222}A_{22}p_2^4\\&+S_{4223333}A_{22}p_3^4+S_{4121112}(A_{12}+A_{21})p_1^3p_2\\&+S_{4121222}(A_{12}+A_{21})p_1p_2^3\\&+S_{4131333}(A_{13}p_1+A_{23}p_2)p_3^3\\&+S_{4311333}(A_{31}p_1+A_{32}p_2)p_3^3.\end{aligned}\tag{9}$$

Not all the coefficients in Eqs. (8) and (9) are unequal, for $S_{21212}=S_{21111}$; $S_{43113}=S_{41212}$; $S_{4221111}=-S_{4111111}$; $S_{4222222}=-S_{4112222}$.

In an earlier paper[4] Akulov's method was carried out, with respect to S_2 only, in hexagonal crystals. The new assumption concerning the variability of V changes several coefficients, so formulae for all the S are here given in Table III, similar in arrangement to Table II. For details of notation consult l.c.[2]

Numerical values for each S have been computed for the particular axial ratio $c=2(6)^{\frac{1}{2}}/3$ appropriate to close-packing of spheres, both for a primitive array and for hexagonal close-packing.

The elastic energy density in cubic crystals is

$$E_e=\tfrac{1}{2}c_{11}\textstyle\sum A_{ii}^2+\tfrac{1}{2}c_{12}\sum A_{ii}A_{jj}+c_{44}\sum A_{ij}^2\tag{10}$$

and in hexagonal crystals is

$$\begin{aligned}E_e&=\tfrac{1}{2}c_{11}(A_{11}^2+A_{22}^2)+\tfrac{1}{2}c_{33}A_{33}^2\\&+\tfrac{1}{2}c_{12}(A_{11}A_{22}+A_{22}A_{11})\\&+\tfrac{1}{2}c_{13}(A_{11}A_{33}+A_{33}A_{11}+A_{22}A_{33}+A_{33}A_{22})\\&+c_{44}(A_{13}^2+A_{31}^2+A_{23}^2+A_{32}^2)\\&+\tfrac{1}{2}(c_{11}-c_{12})(A_{12}^2+A_{21}^2).\end{aligned}\tag{11}$$

The summations in Eq. (10) are to be interpreted like those in preceding equations but c_{11}, c_{12} and c_{44} are the elastic moduli as usually defined with a different rule of summation. This explains the unsymmetrical occurrence of the factor $\frac{1}{2}$ in Eqs. (10), (11).

In the next step we seek tensor components A_{ii}, A_{ij} to minimize E_m+E_0 for various assignments of p_i, p_j, p_k. The work is much simplified by the fact that for directions of $\mathbf{p}$ naturally of interest the crystal and with it the strain tensor $\mathbf{A}$ possess high symmetry. The results are particularly interesting for dilatationless strains. Not only does the number of terms to be considered decrease, but this assumption about $\mathbf{A}$ supports the assumption already made regarding the invariance of P, and is in itself plausible as a first approximation. Experiments usually show small relative changes in volume in applied fields that result in considerable shearing strains. On the theoretical side the enormous molecular field is supposed to be largely affected by small changes in mean interatomic distance. It follows directly that changes in energy resulting from changes in direction of the relatively tiny magnetic fields used in magnetostriction experiments can have little effect on mean interatomic distance, that is, they cannot produce important dilatations. Finally, the assumption that $\sum A_{ii}=0$ makes minimum total energy occur for minimum energy density, a decided simplification.

[4] L. W. McKeehan, Phys. Rev. **43**, 1025–1029 (1933). Eq. (35) should be changed by adding $+H_{1q}$ to the right-hand side.

TABLE III. (Part 1) *Formula for the coefficients of S_{2+} for hexagonal crystals.*

$\eta_1 = \epsilon^3 \Sigma n_r g_1(x)$
$\eta_2 = \epsilon^3 \Sigma n_r r_{12}^2 r^{-2} x^2 g_2(x)$
$\eta_3 = \epsilon^3 \Sigma n_r c^2 r_3^2 r^{-2} x^2 g_2(x)$
$\eta_4 = \epsilon^3 \Sigma n_r r_{12}^4 r^{-4} x^4 g_3(x)$
$\eta_5 = \epsilon^3 \Sigma n_r r_{12}^2 c^2 r_3^2 r^{-4} x^4 g_3(x)$
$\eta_6 = \epsilon^3 \Sigma n_r c^4 r_3^4 r^{-4} x^4 g_3(x)$
$\eta_7 = \epsilon^3 / 3(\pi)^{\frac{1}{2}}$
$\eta_8 = (\pi/c\sqrt{3}) \Sigma f n_q q_{12}^2 q^{-2} \exp(-\pi^2 q^2/\epsilon^2)$
$\eta_9 = (\pi/c\sqrt{3}) \Sigma f n_q c^{-2} q_3^2 q^{-2} \exp(-\pi^2 q^2/\epsilon^2)$
$\eta_{10} = (\pi/c\sqrt{3}) \Sigma f n_q q_{12}^4 q^{-4} (1+\pi^2 q^2/\epsilon^2) \exp(-\pi^2 q^2/\epsilon^2)$
$\eta_{11} = (\pi/c\sqrt{3}) \Sigma f n_q q_{12}^2 c^{-2} q_3^2 q^{-4} (1+\pi^2 q^2/\epsilon^2) \exp(-\pi^2 q^2/\epsilon^2)$
$\eta_{12} = (\pi/c\sqrt{3}) \Sigma f n_q c^{-4} q_3^4 q^{-4} (1+\pi^2 q^2/\epsilon^2) \exp(-\pi^2 q^2/\epsilon^2)$

Coefficients in Eq. (9)

	η_1	η_2	η_3	η_4	η_5	η_6	η_7	η_8	η_9	η_{10}	η_{11}	η_{12}
S_{200}	−1/2	+1/4					+2	−2				
S_{20033}		−1/4	+1/2					+2	−4			
S_{2ii}	+1/2			−1/16			−2	+4		−1		
S_{233}		−1/4	+1/2	+1/16	−1/4					+1	−4	
S_{21111}		+1/2		−1/8				+4		−2		
S_{2ii33}		+1/4	−1/2	+1/16	−1/4			−4	+8	+1	−4	
S_{23333}			+1	−1/16	+1/2	−1/2			+8	−1	+8	−8
S_{21313}			+1		−1/2			+4			−8	
S_{23113}		+1/2			−1/2				+8		−8	

TABLE III. (Part 2) *Formula for the coefficients of S_{4+} for hexagonal crystals.*

$\tau_1 = \epsilon^5 \Sigma n_r g_2(x)$
$\tau_2 = \epsilon^5 \Sigma n_r r_{12}^2 r^{-2} x^2 g_3(x)$
$\tau_3 = \epsilon^5 \Sigma n_r c^2 r_3^2 r^{-2} x^2 g_3(x)$
$\tau_4 = \epsilon^5 \Sigma n_r r_{12}^4 r^{-4} x^4 g_4(x)$
$\tau_5 = \epsilon^5 \Sigma n_r r_{12}^2 c^2 r_3^2 r^{-4} x^4 g_4(x)$
$\tau_6 = \epsilon^5 \Sigma n_r c^4 r_3^4 r^{-4} x^4 g_4(x)$
$\tau_7 = \epsilon^5 \Sigma n_r r_{12}^6 r^{-6} x^6 g_5(x)$
$\tau_8 = \epsilon^5 \Sigma n_r r_1^2 r_2^2 (r_1 - r_2)^2 r^{-6} x^6 g_5(x)$
$\tau_9 = \epsilon^5 \Sigma n_r r_{12}^4 c^2 r_3^2 r^{-6} x^6 g_5(x)$
$\tau_{10} = \epsilon^5 \Sigma n_r r_{12}^2 c^4 r_3^4 r^{-6} x^6 g_5(x)$
$\tau_{11} = \epsilon^5 \Sigma n_r c^6 r_3^6 r^{-6} x^6 g_5(x)$
$\tau_{12} = \epsilon^5 / 5(\pi)^{\frac{1}{2}}$
$\tau_{13} = (\pi^3/c\sqrt{3}) \Sigma f n_q q_{12}^4 q^{-2} \exp(-\pi^2 q^2/\epsilon^2)$
$\tau_{14} = (\pi^3/c\sqrt{3}) \Sigma f n_q q_{12}^2 c^{-2} q_3^2 q^{-2} \exp(-\pi^2 q^2/\epsilon^2)$
$\tau_{15} = (\pi^3/c\sqrt{3}) \Sigma f n_q c^{-4} q_3^4 q^{-2} \exp(-\pi^2 q^2/\epsilon^2)$
$\tau_{16} = (\pi^3/c\sqrt{3}) \Sigma f n_q q_{12}^6 q^{-4} (1+\pi^2 q^2/\epsilon^2) \exp(-\pi^2 q^2/\epsilon^2)$
$\tau_{17} = (\pi^3/c\sqrt{3}) \Sigma f n_q q_1^2 q_2^2 (q_1+q_2)^2 q^{-4} (1+\pi^2 q^2/\epsilon^2) \exp(-\pi^2 q^2/\epsilon^2)$
$\tau_{18} = (\pi^3/c\sqrt{3}) \Sigma f n_q q_{12}^4 c^{-2} q_3^2 q^{-4} (1+\pi^2 q^2/\epsilon^2) \exp(-\pi^2 q^2/\epsilon^2)$
$\tau_{19} = (\pi^3/c\sqrt{3}) \Sigma f n_q q_{12}^2 c^{-4} q_3^4 q^{-4} (1+\pi^2 q^2/\epsilon^2) \exp(-\pi^2 q^2/\epsilon^2)$
$\tau_{20} = (\pi^3/c\sqrt{3}) \Sigma f n_q c^{-6} q_3^6 q^{-4} (1+\pi^2 q^2/\epsilon^2) \exp(-\pi^2 q^2/\epsilon^2)$

Coefficients in Eq. (10)

Subscript of S	τ_1	τ_2	τ_3	τ_4	τ_5	τ_6	τ_7	τ_8	τ_9	τ_{10}	τ_{11}	τ_{12}	τ_{13}	τ_{14}	τ_{15}	τ_{16}	τ_{17}	τ_{18}	τ_{19}	τ_{20}
400	+1/8	−1/8		+1/64								−1	+1/2							
400		−1/80	+1/40	+1/320	−1/80								+1/10	−2/5						
400				+3/2240	−3/280	+1/280							+3/70	−12/35	+4/35					
4ii	−3/8		−1/8	+9/64	+1/8		−1/64		−1/64			+3	−5			+1		+1		
4ii		+1/80	−1/40	+1/64	−1/40	−1/40	−1/320		+3/320	+1/80			−1	+4		+1/5		−3/5	−4/5	
4ii				+3/2240	−3/280	+1/280	−3/2240		+3/320	+1/140	−1/280		−3/7	+24/7	−8/7	+3/35		−3/5	−16/35	+8/35
433	+1/4	+1/8		−5/32			+1/64					−2	+4			−1				
433			+1/20	−3/160	+1/40		+1/320		−1/80				+4/5	−12/5		−1/5		+4/5		
433				−3/1120		+1/140	+3/2240		−3/280	+1/280			+12/35	−72/35	+16/35	−3/35		+24/35	−8/35	
41111	+1/4	−3/16	+1/8	−1/32			+3/256	−27/512	−1/64			−2	+3	−4		−1/4	−8	+1		
41122	+1/4	+1/16	+1/8	−3/32	−1/4		+3/256	−27/512	+3/64			−2	+3	+4		−1/4	−8	−3		
41133	+1/4	+1/16	+1/8		−1/4	−1/4	−3/256	+27/512	+3/64	+1/8		−2	−3	+28		+1/4	+8	−3	−8	
42211	+1/4	+1/16	+1/8	−3/32	−1/4		+1/256	+27/512	+3/64			−2	+3	+4		−3/4	+8	−3		
42222	+1/4	−3/16	+1/8	−1/32			+1/256	+27/512	−1/64			−2	+3	−4		−3/4	+8	+1		
42233	+1/4	+1/16	+1/8		−1/4	−1/4	−1/256	−27/512	+3/64	+1/8		−2	−3	+28		+3/4	−8	−3	−8	
41212		−1/4		+1/16	+1/4				−1/16					−8				+4		
41313			−1/2		+1/2				−1/16				−2					+4		
4111111				+1/32	−1/8		−1/96	+9/128	+1/32				−1	+4			+32/3	−2		
4112222				−1/32	+1/8			+9/128	−1/32				+1	−4		−2/3	+32/3	+2		
4113333					+1/8	−1/12	+3/256	−27/512	−3/64	−5/48	+1/24		+3	−28	+32/3	−1/4	−8	+3	+20/3	−8/3
4223333					+1/8	−1/12	+1/256	+27/512	−3/64	−5/48	+1/24		+3	−28	+32/3	−3/4	+8	+3	+20/3	−8/3
4121112				+1/16	−1/4		−1/192	−9/128	+1/16				−2	+8		+1	−32/3	−4		
4121222				+1/16	−1/4		−1/64	+9/128	+1/16				−2	+8		+1/3	+32/3	−4		
4131333					−1/4	+1/6			+1/16	−1/12			+2	−8				−4	+16/3	
4311333				−1/16	+1/4				+1/16	−1/12				+8	−16/3			−4	+16/3	

Table V shows what strain components still need to be computed for several interesting directions of **p** in cubic and hexagonal crystals. The last column in Table V, headed "Cases," refers for comparison to items in a table given by Bitter[5] in an analysis of the experimental material.

The strain components in lines 2 and 3 of

[5] F. Bitter, *Introduction to Ferromagnetism* (1937), pp. 253–254.

TABLE IV. *Hexagonal arrangements.*

Axial ratio $c=2/3(6)^{\frac{1}{2}}=1.6330$.

SUBSCRIPT OF S	NUMERICAL PRIMITIVE	VALUE OF S CLOSE-PACKED	SUBSCRIPT OF S	NUMERICAL PRIMITIVE	VALUE OF S CLOSE-PACKED
200	2.75741	2.96023	400	0.95461	0.33815
20033	−3.82936	0.00507	4*ii*	−7.6369	−2.7052
2*ii*	−4.8288	−4.8920	433	6.6382	5.6646
233	2.0848	0.7779	41111	35.1298	29.6909
21111	−4.1428	−3.8634	41122	16.2140	9.7378
2*ii*33	9.7435	0.7678	41133	−60.2145	−18.6979
23333	−10.3972	−6.1972	42211	−26.4942	−29.9450
21313	0.0268	−2.3076	42222	−7.5784	−9.9920
23113	7.7123	−2.3177	42233	−17.5062	20.9850
			41212	18.9158	19.9530
			41313	−0.1765	13.1900
			4111111	−39.5064	−38.0945
			4112222	−17.4380	−14.8160
			4113333	71.7286	18.3590
			4223333	29.0204	−21.3238
			4121112	6.4038	3.1767
			4121222	−50.5406	−49.7338
			4131333	0.4117	−30.7766
			4311333	−44.1369	−46.5571

Table V may be simplified by referring them to the principal axes of the strains. In line 2 this gives $A[110]=A_{11}+A_{12}$; $A[1\bar{1}0]=A_{11}-A_{12}$; $A[001]=-2A_{11}$. In line 3 it gives $A[111]=2A_{23}$; $A[1\bar{1}0]=-A_{23}$; $A[11\bar{2}]=-A_{23}$.

In applying the formulae here derived for E_m to iron and nickel the values of N, P, a, F_4 given in Table III of l.c.[2] are used. In case of cobalt a slight improvement over l.c.[2] is introduced by making $a=2.5024\times10^{-8}$ cm, instead of using the experimental value, $a=2.507\times10^{-8}$. This changes ρ enough to raise F_4 at $\lambda=\pi/2$ from 0.9090 to 0.9124. The calculated value of ${}_0E_m$ now corresponds to a fictitious crystal having precisely the axial ratio $c=2(6)^{\frac{1}{2}}/3$ used in computing Table IV. The strain **A′** necessary to change this fictitious crystal to the actual strain-free cobalt crystal, with $c=1.624$, $a=2.507\times10^{-8}$ cm, is dilatationless, as is also the further strain **A″** necessary to minimize (E_m+E_e). The strains **A′** and **A″** both affect E_m but only the strain **A″** (the magnetostrain) affects E_e. The value of E_m under **A′** alone will be denoted by E_m'.

TABLE VI. *Elastic moduli* $(erg\cdot cm^{-3})$.

METAL	$10^{-12}\times$ c_{11}	c_{12}	c_{44}
Iron[6]	2.369	1.406	1.160
Nickel	3.42	1.98	0.789
Cobalt	3.47	1.95	0.76

The computation of E_e requires values for elastic moduli appropriate to single crystals, not all of which are as yet known. In case of iron the experiments of Goens and Schmid[6] yield all needed data. In the case of nickel new (unpublished) values were furnished by Quimby, who has supplemented the work of Zacharias[7] on a single crystal with measurements of Young's modulus and shear modulus on a polycrystalline rod like that used by Siegel and Quimby[8] in studies on Young's modulus only. In the case of cobalt there seem to be no accepted values of moduli for use in Eq. (12). Making the very daring assumptions that cobalt is elastically isotropic, and that it is otherwise indistinguishable from hickel allows using Quimby's newly measured moduli for polycrystalline nickel together with the added conditions for elastic isotropy. About the only justification for these assumptions is the fact that Young's moduli[9] for nickel and cobalt and the compressibilities as well[10] are nearly indistinguishable in polycrys-

[6] E. Goens and E. Schmid, Naturwiss. **19**, 520–524 (1931).
[7] J. Zacharias, Phys. Rev. **44**, 116–122 (1933).
[8] S. Siegel and S. L. Quimby, Phys. Rev. **49**, 663–670 (1936).
[9] J. Zacharias, reference 7; K. Nakamura, Sci. Rep. Tohoku Imp. Univ. Honda Volume, 146–157 (1936).
[10] P. W. Bridgman, Proc. Am. Acad. **58**, 163–242 (1923).

TABLE V. *Characteristics of magnetostrictive strains.*

CRYSTAL SYMMETRY	INDICES OF **p**	COMPONENTS OF **p** p_1	p_2	p_3	STRAIN COMPONENTS A_{11}	A_{22}	A_{33}	A_{23} AND A_{32}	A_{31} AND A_{13}	A_{12} AND A_{21}	CASES[5]
Cubic	[100]	1	0	0	A_{11}	$-A_{11}/2$	$-A_{11}/2$	0	0	0	*a*, *b*, *c*
	[110]	$1/\sqrt{2}$	$1/\sqrt{2}$	0	A_{11}	A_{11}	$-2A_{11}$	0	0	A_{12}	*d*, *e*, *f*
	[111]	$1/\sqrt{3}$	$1/\sqrt{3}$	$1/\sqrt{3}$	0	0	0	A_{23}	A_{23}	A_{23}	*g*, *h*
Hexagonal	[001]	0	0	1	A_{11}	A_{11}	$-2A_{11}$	0	0	0	
	[100]	1	0	0	A_{11}	A_{22}	A_{33}	0	0	0	
	[120]	0	1	0	A_{11}	A_{22}	A_{33}	0	0	0	

TABLE VII. *Calculated strains, and energy densities in erg·cm^{-3}.*

METAL		$10^6\times$				$10^{-6}\times$	
	p	A_{11}	A_{22}	A_{33}	A_{23}	${}_0E_m$	E_e
Iron	[100]	5.62	−2.81	−2.81	—	−8.5037	−0.000023
	[110]	−4.17	−4.17	8.34	−6.01	−5.9381	−0.000134
	[111]	—	—	—	1.26	−5.0829	−0.000011
Nickel	[100]	0.70	−0.35	−0.35	—	−0.3974	−0.0000005
	[110]	−0.51	−0.51	1.02	−0.95	−0.6667	−0.0000026
	[111]	—	—	—	0.30	−0.7564	−0.0000004
Cobalt*	[001]	−2.73	−2.73	+5.46	—	−5.3933	−0.0000340
	[100]	−11.48	+6.44	+5.04	—	−4.6513	−0.0001509
	[120]	−4.97	−0.07	+5.04	—	−4.6513	−0.0000381

* Corrected for preliminary strain: $A_{11}'=0.001839$, $A_{22}'=0.001839$, $A_{33}'=-0.003678$, to make $c=1.624$, and $a=2.507\times10^{-8}$ cm.

talline specimens. The assumption of isotropy leaves only two independent moduli, c_{11} and c_{12}, instead of the five appearing in Eq. (12), since $c_{33}=c_{11}$ and $c_{44}=(c_{11}-c_{12})/2$.

The moduli used in the final calculations are collected in Table VI. The strain components for minimum (E_m+E_e) are presented in Table VII, together with values of ${}_0E_m$ (or ${}_0E_m'$), the energy density in unstrained crystals for the same directions of **p**, and the energy density increase, $(E_m-{}_0E_m+E_e)=-E_e$, arising from magnetostrain. It is obvious that the energy densities associated with magnetostriction are trivial, and this remains true whether the theoretical magnetostrains of Table VII or experimental values[5] are used. This justifies the principal results of l.c.[2] which ignored strain effects altogether.

The calculated magnetostrains in iron are much nearer to experimental values than are those in nickel. The signs of all components agree with the single exception of $A[001]$ for **p** along [110]. In nickel the calculated values are much too low in absolute magnitude and do not exhibit the approximate isotropy, both of longitudinal contraction and of transverse expansions, so striking in the experiments. In cobalt the calculated values are again too small and do not give all the details of the experiments[11] with which they must be compared. The fit, such as it is, is almost certainly fortuitous, in view of the roughness with which the elastic moduli have been estimated. Better data are badly needed.

The inadequacy of purely magnetic interactions to account for magnetostriction in nickel suggests that the nonmagnetic interactions, collectively accounting for the molecular field, do not produce mere dilatation (here contraction in volume) as hitherto supposed.

[11] Z. Nishiyama, Sci. Rep. Tohoku Imp. Univ. **20**, 323–341 (1931).

FEBRUARY 15, 1938 PHYSICAL REVIEW VOLUME 53

Magnetic Interaction in Pyrrhotite and in Magnetite

L. W. MCKEEHAN
Sloane Physics Laboratory, Yale University, New Haven, Connecticut

(Received January 3, 1938)

Magnetic potential energy densities, including quadrupole terms, have been computed for important directions of magnetization in pyrrhotite and in magnetite. In pyrrhotite the results are in contradiction with experiment, suggesting that other causes of ferromagnetic anisotropy greatly predominate. In magnetite the results suggest that all the iron atoms contribute to the ferromagnetism, neither ferric nor ferrous iron being adequate separately to explain the observed behavior. With this assumption the results are in qualitative agreement with experiment, calculated anisotropy being greater than that reported in the literature of the subject.