EXTENSION AND PERIOD OF A CONICALLY WOUND SPRING

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Abstract

By imposing certain restrictions on the shape of a conically wound spring, it is valid to assume that the extension under a load is due entirely to torsion. It is found that the extension is then given to an approximation by the expression:

$$Z=\frac{2MgR^2L}{3\pi n r_1^4}$$

Mg is the load, R is the radius of the end coil, L and r_1 are respectively the length and radius of the wire of which the spring is made, and n is its rigidity modulus. For a truncated conically wound spring, this expression becomes:

$$Z=\frac{2MgN(R+r)(R^2+Rr+r^2)}{3nr_1^4}$$

N is the number of coils of the spring, and r is the radius of the small end coil. By making certain assumptions the period is shown to be:

$$T=2\pi\left(\left(1+\frac{xm}{M}\right)\frac{z}{g}\right)^{1/2}$$

m is the mass of the spring, and X is a function of the ratio of the radii of the two end coils.

Introduction

The conically wound spring is in quite general use, but it does not lend itself so readily to theoretical treatment as does the helical. Consequently, while the theory of the helical spring is well developed,[1] the conically wound spring presents a field for investigation which is practically untouched. If this type of spring is defined as a uniform wire wound in such a way that the elastic central-line lies upon a right cone and cuts all the generators under the same angle, the equations

$$X=ce^{au}\cos u \qquad Y=ce^{au}\sin u$$
$$Z=be^{au}$$

where a, b, and c, are constants, and u is the general parameter, may be taken to represent the elastic central-line. These equations offer one method

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[1] Kelvin and Tait. "Treatise on Natural Philosophy," part II, pp. 139–144.

of solution to the problem of the conically wound spring, but a less tedious method will be presented here.

Extension of a Conical Spring

The quantity of stored energy in a twisted straight rod is given by the expression:

$$Q=\tfrac{1}{4}\pi n r_1^4(d\theta/ds)^2 l$$

l is the length of the rod, $d\theta/ds$ is the rate of twist, r_1 is the radius of the rod, and n is its rigidity modulus. Consider a small element ds of a conical spring, i.e. a conically wound spring which terminates at the apex of the cone. It is evident that since the twisting torque is a variable, the rate of twist will be a variable. Hence the stored energy of this particular element is:

$$dQ=\tfrac{1}{4}\pi n r_1^4(d\theta/ds)^2 ds$$

provided the element ds is normal at all times to the plane of the line of action of the force due to the load and the radius of the coil. In the development of the equations of the elastic central-line, this restriction may be shown to lead to the result that

$$a=R/L \tag{1}$$

R is the radius of the end coil, and L is the length of the spring measured along the wire. The expression for the torsional couple on a rod or wire of circular cross section is:

$$Mga_1=\tfrac{1}{2}\pi n r_1^4 d\theta/ds$$

a_1 is the arm of the couple, r_1 is the radius of the rod, n is its rigidity modulus, and $d\theta/ds$ is, as above, the rate of twist. Utilizing the last three equations, and integrating between the limits L and O, there results:

$$Q=\frac{(Mg)^2R^2L}{3\pi n r_1^4}.$$

If Z is the extension the spring undergoes, it follows from Hook's law that $Q=\frac{1}{2}MgZ$, and hence

$$Z=\frac{2MgR^2L}{3\pi n r_1^4}. \tag{2}$$

The corresponding expression for the helical spring is:

$$Z=\frac{2MgR^2L}{\pi n r_1^4}.$$

Comparing these two equations, the following relationship between the helical and conical spring may be noted: When a given weight is applied, the extension of a conical spring is equal to one-third the extension of a helical spring made of wire of the same dimensions and the same modulus of rigidity, the helical spring having a radius equal to the radius of the end coil of the conical spring.

Modification for Truncated Conically Wound Spring

It is now comparatively easy to modify the above result to fit the case of a truncated conically wound spring which has a coil of radius r at the upper end and a radius R at the lower end. Imagine the wire of a truncated conically wound spring to be extended upward to the apex of the cone so that it becomes a spring whose extension is given by Eq. (2). Now by using the relationship (1), and by a simple composition of known extensions given by Eq. (2), the result for the truncated conically wound spring is found to be:

$$Z=\frac{2MgL(R^2+Rr+r^2)}{3\pi n r_1^4}.$$

Or, since L is in general difficult of measurement, we may approximate it and write[2]

$$Z=\frac{2MgN(R+r)(R^2+Rr+r^2)}{3nr_1^4} \tag{3}$$

N is the number of turns of wire in the coil.

Period of Conically Wound Spring

From Eqs. (1) and (2), it is evident that the velocity of any element along the wire of a conical spring may be expressed as $2Mga^2S^3/3\pi n r_1^4$, where S is the distance of the element measured along the wire. The fact that the velocity of the element of a conical spring varies in this way with the cube of the distance along the wire, can be used to determine the expression for the kinetic energy of the truncated spring when vibrating with a load. However, in order to apply this principle to the truncated spring, it is necessary as in the calculation of the extension to imagine the truncated spring as an integral part of a larger conical spring. Then by a simple composition of extensions, and by a determination of the dimensions of the hypothetical part of the conical spring in terms of the known dimensions of the truncated spring, it is possible to work back to the real case. This method gives for the kinetic energy of the truncated conically wound spring:

$$E=\frac{1}{2}m\dot{Z}^2\left[\frac{2+6K+12K^2+13K^3+9K^4}{14(1+K+K^2)^2}\right]$$

$\dot{Z}$ is the instantaneous velocity of the vibrating load, m is the mass of the spring, and K is the ratio, r/R. Assuming that the system is conservative, the expression for the period is found to be:

[2] In the October, 1928, issue of the J.O.S.A. & R.S.I., R. B. Abbott developes by another method a different expression, namely $Z=MgN(R+r)(R^2+r^2)/nr_1^4$. This expression does not agree so well with experimental data now available as does the one developed in this paper.

$$T=2\pi\left(\left(1+\frac{xm}{M}\right)\frac{z}{g}\right)^{1/2}$$

where X is the value of the fraction of mass of the spring which enters into the expression for its kinetic energy. The calculated values which this fraction assumes for various values of K are given below.

TABLE I.

K	Fraction	K	Fraction	K	Fraction	K	Fraction
0	0.1429	0.25	0.1861	0.55	0.2480	0.80	0.2978
0.05	.1503	.30	.1961	.60	.2583	.85	.3071
.10	.1585	.35	.2063	.65	.2685	.90	.3161
.15	.1673	.40	.2167	.70	.2785	.95	.3249
.20	.1764	.45	.2271	.75	.2883	1.00	.3333
		.50	.2376				

Equation (3) has been checked experimentally with good agreement. Using a small phosphor-bronze spring, the values found were:

$$Mg=980\times 2.392 \text{ dynes}$$
$$N=112.5 \text{ turns}$$
$$R=0.9115 \text{ cm}$$
$$r=0.4115 \text{ cm}$$
$$r_1=0.026 \text{ cm}$$
$$n=4.36\times 10^{11} \text{ dynes/cm}^2$$

From direct measurement, the extension was found to be 1.60 cm. The value calculated by the above data gives 1.605 cm, an error of 0.3 percent, which is within the limits of experimental error. Using the same spring with a load of 9.955 grams, an experimental value of 6.70 cm was found for the extension, as compared with 6.682 cm by formula, an error of less than 0.3 percent.