AN OPTICAL DETERMINATION OF AXIAL STRESSES IN LONG RECTANGULAR PLATES UNDER TORSION

By Isay A. Balinkin

Abstract

The use of polarized light is extended to the determination of the distribution and evaluation of axial stresses in long rectangular plates subjected to torsion. The equation connecting the stresses, σ, with the distance from the axis of rotation, r is

$$\sigma = \frac{1}{2}\frac{E\varphi^2}{l^2}\left(r^2 - \frac{h^2}{12}\right),$$

from which it follows that the compression along the central axis is half as large as the tension along the edges. A new photometric method for measuring the stress intensities was employed. As a photometer, a selenium cell with a narrow window was used; and it is suggested that the photo-electric cell may be applied to this purpose with great advantage.

Introduction

A PHOTO-ELASTIC method of determining stresses and their distribution can be successfully used only in case of "two-dimensional" problems. Although the actual problems in elasticity involve three dimensional bodies, it is possible in certain cases to investigate the stress distribution by means of polarized light, as if the stresses were in one plane.

In order to be able to reduce a problem by one dimension we have to assume that the stresses along the direction of the polarized ray do not affect the state of polarization. Also, the type of problem must be such that either the stresses are the same in all vertical parallel planes or for the case of "generalized plane stresses," they are a known function of the distance.

The relative retardation of the two trains of waves, into which a plane polarized beam is decomposed when passing through a transparent specimen under stress, is expressed by

$$R = C\int_0^t (P-Q)dt \tag{1}$$

where $(P-Q)$ is the difference in the principal stress intensities in the plane perpendicular to the ray of light, C is a constant depending upon the optical characteristics of the material, and t is the thickness of the specimen. It is important to notice from Eq. (1) that stresses along the axis of the propagated ray have no effect on R which is the relative retardation of the two trains of waves with their corresponding electric and magnetic vectors perpendicular to one another.

It would be very desirable to extend the photo-elastic method to problems involving torsion. This field is in great need of an experimental means for determining shearing components since, although mathematical analysis provides solutions for a number of cases, yet they are limited to the consideration of infinitesimal strains.

The present paper takes up the question of evaluating, by the use of polarized light, the axial stresses produced in long rectangular plates which are subjected to torsion. When a beam of light passes through a plate subjected to torsion and introduced between crossed Nicols, two black parallel lines appear on the screen indicating the position of the neutral axes. These two neutral axes are due to the secondary axial tensional and compressional stresses parallel to the axis of twist. There is also one neutral axis passing through the central points in the cross-sections where the shear stresses are absent. Thus, in a long rectangular plate under torsion, we have to distinguish one neutral axes passing through the points of zero shear, and the other two passing through the points of zero axial stresses in the plane of the twisted plate.

Analytical Theory

When a couple M is applied to one end of a long plate, a shearing strain is created within the body, involving two simple shears at each point. One of these shears is in the plane perpendicular to the axis of rotation and the

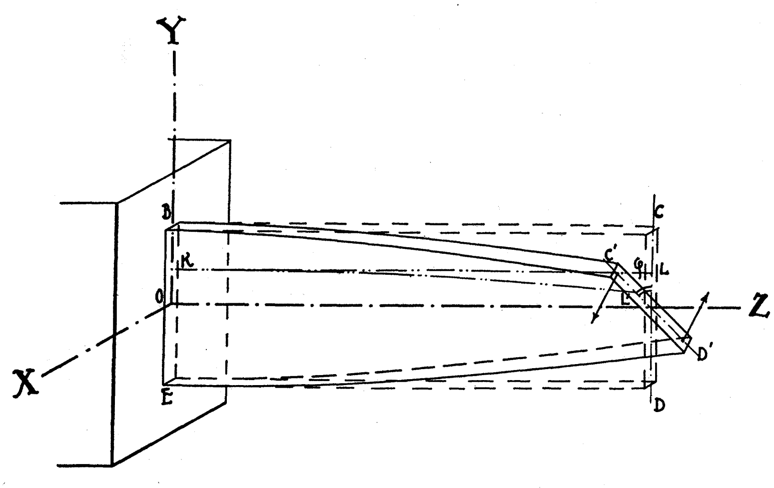


Fig. 1. A long rectangular plate under the action of a turning moment.

other is a longitudional displacement along the centarl axis. The latter produces a distorsion in the non-circular cross-sectional planes.

The following is essentially C. Weber's[1] derivation of axial stresses in terms of the modulus of elasticity E, angle of twist ϕ, and the dimensions of the plate.

When a plate, as in Fig. 1, is twisted through an angle ϕ from the position $BCDE$ into $BC'D'E$ a straight line element, KL, of length l at a distance r

[1] "Die Lehre der Drehungsfestigkeit" Forschungsarbeit Heft 249 des Vereines Deutscher Ingenieure (V. D. I).

from the Z axis is changed into a helix whose length is given by

$$s=(l^2+r^2\phi^2)^{1/2}$$

The increase in length is

$$\Delta l=(l^2+r^2\phi^2)^{1/2}-l$$

and, on expanding the first term and neglecting the higher powers of ϕ and r, we obtain

$$\Delta l=r^2\phi^2/2l$$

The strain due to this elongation is $r^2\phi^2/2l^2$ and therefore the stress

$$\sigma'=E\Delta l/l=Er^2\phi^2/2l^2$$

The total force necessary to retain the elements of the surface $C'D'$ at their original length l from the XY plane is

$$Z'=\int(Er^2\phi^2dA/2l^2)$$

where A is the area of the rectangular section CD. Evaluating this force we get

$$Z'=(E\phi^2/2l^2)\int_{-b/2}^{+b/2}\int_{-h/2}^{+h/2}(x^2+y^2)dx\,dy$$
$$=E\phi^2bh(b^2+h^2)/24l^2$$

Since there is no external force acting at the end of the plate, the stress produced by Z' will be equally distributed over the cross-section bh and we can compute its intensity

$$\sigma''=-Z'/\int dA=-E\phi^2(b^2+h^2)/24l^2$$

Evidently, the final stress intensity parallel to the axis is the summation of σ' and σ''

$$\sigma=\sigma'+\sigma''=\frac{1}{2}\ \frac{Er^2\phi^2}{l^2}-\frac{1}{24}\ \frac{E\phi^2}{l^2}(b^2+h^2)$$
$$\sigma=\frac{1}{2}\ \frac{E\phi^2}{l^2}\left(r^2-\frac{b^2+h^2}{12}\right).$$

Considering σ as a function of r, we can readily see that $\sigma=0$ when

$$r=r_0=\pm\left[(b^2+h^2)/12\right]^{1/2}$$

Thus, at a distance r_0 on either side of the axis Z, the axial stress falls to zero and we have two neutral axes. When r is less than r_0, σ is negative and the central part of the plate is in the state of compression; similarly, places in the plate where r is larger than r_0 must be in a state of tension.

The maximum tension and compression are

$$\sigma(\text{max. compr.}) = E\phi^2 h^2/24l^2$$

$$\sigma(\text{max. tens.}) = E\phi^2 h^2/12l^2$$

Thus, the maximum tension at the edges of the twisted plate is twice as large as the maximum compression along the z axis and this ratio is independent of the angle of twist, ϕ.

Description of the Apparatus and the Experimental Methods

The experimental proof of the preceding mathematical analysis is shown by the following demonstration. Two Nicol prisms, $2'' \times 1\frac{1}{2}'' \times 5''$ with their principal planes at right angles, were mounted on an optical bench and a "half wave-length plate" method was used to set them exactly at 90°.[2]

A 1000-watt Mazda lamp was used as a light source. On account of the photometric measurements of stresses, it was of considerable importance to maintain the light source at constant intensity. The lamp was connected to a d.c. storage battery, and with a regulating resistance in series, it was possible to keep the voltage constant within one tenth of a volt.

A specimen plate of transparent celluloid, Fig. 2, was fixed between the parallel jaws of two chucks A and B, of which A was stationary and B

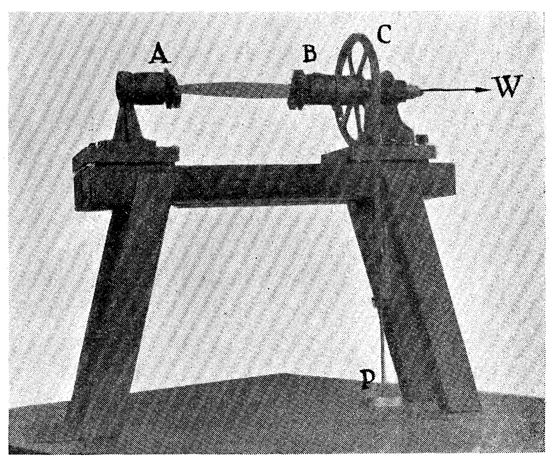


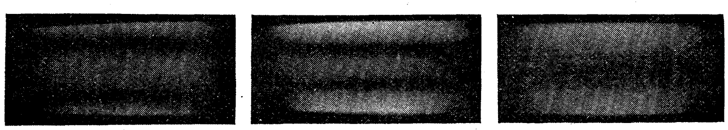
Fig. 2. Apparatus for twisting and examining long plates between crossed Nicols.

could be rotated through an angle measured by a divided circle, C. The first chuck was set back at an angle equal to one half of the final angle of twist so as to leave the central part of the specimen perpendicular to the direction of the ray.

A preliminary investigation of the compressional stresses along the Z axis was first carried out by applying a tensional force, W, and measuring the distances between the two neutral axes. The head A was permanently connected to the table, while B was placed on smooth rollers permitting free motion along the axis. The effect of applying an axial tension was to reduce the compression in the central part of the plate, and thus shift the two neutral axes nearer to the center. When they were made to coincide the stress due to the outside load, *W*, was equal to the maximum axial compression. This is clearly shown in Fig. 3.

This method of determining the distribution of axial stresses could be extended also to the part of the plate in tension, by introducing a separate tension piece perpendicular to the axis. It proved, however, to be difficult to record exactly the distances between the neutral axes, since they were not sharply defined. A purely photometric method of measuring the stress distribution was devised and applied successfully to this problem.

When a tensional stress is exerted on a plate placed between two crossed Nichols the field, originally dark, grows brighter with an increasing stress. This suggests that the measure of the light intensity could be utilized for the measurement of the stresses. In the first place a calibration curve was



(*a*) (*b*) (*c*)

Fig. 3. (a) Two neutral axes in a long plate under torsion, when examined between crossed Nicols. (b) The distance between the axes decreases as tension is applied at the ends. (c) One neutral axis is shown when the outside tension is equal to the compression in the central part of the specimen.

constructed, by plotting the light intensities against the measured stress intensities which were applied to the test piece.

A photometric instrument suitable for this purpose must be able to explore a very limited field at a time and have also a comparatively high sensitivity, especially for a low illumination. A McWilliams' selenium cell[3] was adopted for these measurements. The cell was covered completely with black paper leaving only a narrow horizontal slit of 2 mm width. An indicator, attached at the same level as the slit, pointed on the screen the exact place at which the intensity was measured. The cell, mounted on a cathetometer, could be easily set at any desired position in the projection of the specimen on the screen. The resistance of the selenium cell at different positions along the vertical line was measured by means of a Wheatstone bridge. The effect of fatigue was eliminated by taking the mean of the two readings while the load was gradually increased and then decreased.

Calibration Curve

The calibration curve will have to show the relation existing between the intensity of stress and illumination, or in the case of a selenium cell it will be a relation between the stress intensity and the resistance of the cell.

[2] A. Schuster, Theory of Optics, 123, 3rd edition.

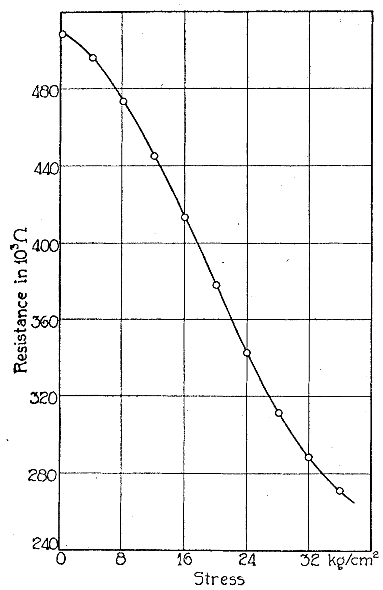
[3] McWilliams, Photo-electric cells and amplifying relays catalog number two.

All specimens, including test pieces for the calibration curve, were cut from the same sheet of celluloid in order to assure the uniformity of their elastic characteristics. Acknowledgments and thanks are due to the Du Pont Viscoloid Co. for the supply of remarkably transparent celluloid without initial stresses.

A narrow plate of celluloid was firmly fastened in the chucks A and B, an axial pull applied to it, and the resistance of the cell measured. In the range of the variation of the stress between 0 and 40 kg/cm², it was found that the corresponding variation in the resistance of the cell due to the change in the intensity of illumination was from 510×10^3 to 250×10^3 ohms. From these data a stress-resistance curve was plotted, as shown in Fig. 4.

Results

The curve obtained for the stress distribution will be given only for one case which is taken as the average of a number of plates investigated by the method described in this paper. The dimensions of the specimen were as follows: length between the jaws $l=14.4$ cm, width $h=1.8$ cm, thickness $b=.195$ cm. Young's modulus $E=21650$ kg/cm², and angle of twist $\phi=60°$.



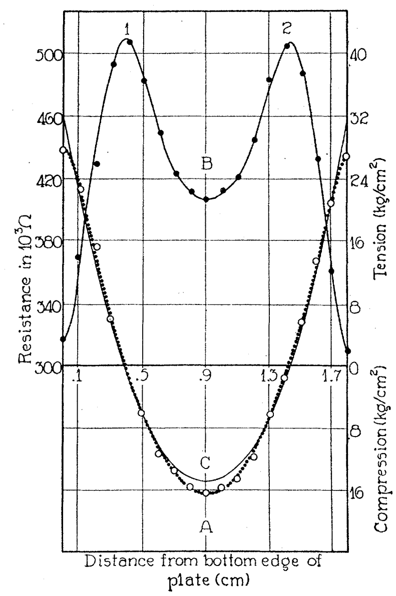


Fig. 4. Stress-resistance calibration curve.

Fig. 5. Curve A represents the axial stresses as determined from curve B by the use of the stress-resistance calibration curve in Fig. 4. Curve C represents the computed stresses.

The curve, B in Fig. 5 gives the values of the resistance of the cell, when its slit was placed at different positions along the width of the plate. Points 1 and 2 of maximum resistance correspond to the two black neutral lines in Fig. 3. Referring back to the calibration curve, Fig. 4, and changing the units of the ordinates into kg/cm² we can plot in the same coordinate plane the actual distribution of stresses, which gives the curve A.

When constants for E, ϕ, l, and h are substituted in the equation

$$\sigma=\frac{1}{2}\,\frac{E\phi^2}{l^2}\left(r^2-\frac{h^2}{12}\right),$$

we can plot the relation between the distance r and the axial stress σ. This is shown on the same Fig. 5 by the curve C.

Conclusions

The axial stress distribution as determined by the photo-elastic method shows a close agreement with the stresses calculated by mathematical analysis. This speaks well for the rigor of the latter, and it certainly must increase our confidence in the photo-elastic method of stress determination. The photometric evaluation of the stress intensities was successfully performed by the use of the selenium cell, although some difficulty was experienced because of the effect of fatigue. This difficulty could be overcome by the use of a photo-electric cell.

It is the author's pleasure to express his grateful thanks to Dr. L. T. More and Dr. R. C. Gowdy for guidance received throughout the course of this investigation.

University of Cincinnati,
Graduate School, Physics Department,
July, 1927.

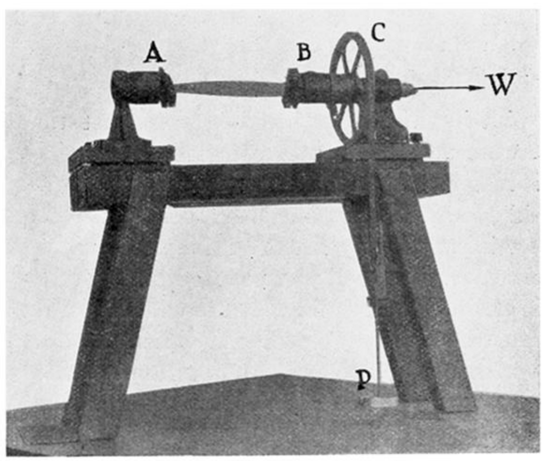
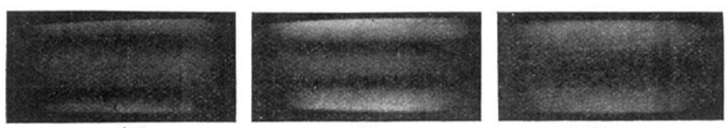


Fig. 2. Apparatus for twisting and examining long plates between crossed Nicols.



(a) (b) (c)

Fig. 3. (a) Two neutral axes in a long plate under torsion, when examined between crossed Nicols. (b) The distance between the axes decreases as tension is applied at the ends. (c) One neutral axis is shown when the outside tension is equal to the compression in the central part of the specimen.