NOTE ON THE EVALUATION OF THE CONSTANT C_2 IN PLANCK'S RADIATION EQUATION.

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THE methods hitherto used[1] for the evaluation of C_2 in Planck's equation

$$J = \frac{c_1}{\lambda^5} \frac{1}{e^{C_2/\lambda\theta} - 1} \tag{1}$$

depend essentially upon graphical processes, *i. e.*, upon processes whose precision is limited by conditions entirely independent of those which control the accuracy of the experimental data. It seems therefore worth while to point out a purely numerical method by means of which C_2 may be found from the experimental results without the necessity of recourse to such graphical methods.

In the present note such a method is developed. The problem is reduced to the solution of an equation of the form

$$u^n - u + A = 0,$$

where n is a positive quantity greater than unity but not necessarily an integer. This equation may be solved by any one of several well-known methods. The details of one of these methods are, for completeness, indicated, and the applicability of the process tested by applying it to data computed by means of equation (1). It is found that the assumed value of C_2 is reproduced with all the precision of which the logarithmic tables used are capable.

Reduction of the Equation.—Let J_1, λ_1, θ_1; J_2, λ_2, θ_2 be two sets of observed corresponding values of J, λ and θ. Let $\lambda_2\theta_2 > \lambda_1\theta_1$. Then

$$J_1 = \frac{c_1}{\lambda_1^5} \frac{1}{e^{C_2/\lambda_1\theta_1} - 1}, \quad J_2 = \frac{c_1}{\lambda_1^5} \frac{1}{e^{C_2/\lambda_2\theta_2} - 1}. \tag{2}$$

Hence

$$\frac{e^{C_2/\lambda_1\theta_1} - 1}{e^{C_2/\lambda_2\theta_2} - 1} = \frac{J_2\lambda_2^5}{J_1\lambda_1^5}. \tag{3}$$

Put now

$$n = \frac{\lambda_2\theta_2}{\lambda_1\theta_1},$$

[1] Cf. Buckingham and Dellinger, Bulletin, Bureau of Standards, 7, 393–406, 1911.

$$x = e^{C_2/\lambda_2\theta_2},$$

$$C = \frac{J_2\lambda_2^5}{J_1\lambda_1^5}.$$

Then since $\lambda_2\theta_2 > \lambda_1\theta_1 > 0$, and $C_2 > 0$; $n > 1$, $x > 1$, $C > 1$. Equation (3) now becomes

$$\frac{x^n - 1}{x - 1} = C,$$

or

(4) $$x^n - Cx + C - 1 = 0.$$

This is an equation of the well-known "trinomial form" and may be solved, for example, by Newton's Method, by the method of Gaussian logarithms, or by iteration.[1]

If we put further

$$u = \frac{x}{C^{1/n-1}} = \frac{e^{C_2/\lambda_2\theta_2}}{C^{1/n-1}},$$

whence

(5) $$C_2 = \frac{\lambda_2\theta_2}{\log_{10} e}\left(\frac{1}{n-1}\log_{10} C + \log_{10} u\right),$$

and

$$A = \frac{C-1}{C^{n/n-1}}, \quad 0 < A < 1,$$

equation (4) becomes

(6) $$u^n - u + A = 0.$$

Equation (4) has two and only two positive roots. One is the trivial value $x = 1$ corresponding to $C_2 = 0$, the other is greater than unity and corresponds to the value of C_2 sought. Hence equation (6) has two positive roots of which the greater gives the desired value of C_2. It is easily seen that both roots of (6) lie between zero and unity. Write now equation (6) in the form

(7) $$u = (u - A)^{1/n}.$$

If we consider the curves

$$y = u, \quad z = (u - A)^{1/n}$$

and note that $y(A) = A > z(A) = 0$, while $y(1) = 1 > z(1) = (1 - A)^{1/n}$ we see that in the neighborhood of the greater root of (6)

$$1 = \frac{dy}{du} > \frac{dz}{du} > 0.$$

Hence[2] if by substitution in (6) we find a rough approximation u_0

[1] Cf. Runge, Praxis der Gleichungen, pp. 52, 141, 81; and especially p. 147, where an equation of form (4) arising from a financial problem is solved by the method of Gaussian logarithms.

[2] Cf. Osgood, Calculus, p. 403.

to the greater root the successive approximations given by

$$
\begin{aligned}
u_1 &= (u_0 - A)^{1/n},\\
u_2 &= (u_1 - A)^{1/n},\\
&\cdots\cdots\cdots\\
u_{m+1} &= (u_m - A)^{1/n},
\end{aligned}
$$

will converge to the desired root u. When this value has been found C_2 is given by

$$C_2 = \frac{\lambda_2\theta_2}{\log_{10} e}\left(\frac{1}{n-1}\log_{10} C + \log_{10} u\right).$$

Numerical Example of the Method.—A test of the above process *as a method of computation* is best obtained by applying it to values of J computed from (1) with assumed values of C_1 and C_2 and noting the precision and rapidity with which the assumed value of C_2 is reproduced.

The following data were assumed:

$$C_1 = 5.29 \times 10^5,\ C_2 = 1.46 \times 10^4,\ \lambda_1 = 2,\ \theta_1 = 800,$$

$\lambda_2 = 5$, $\theta_2 = 1{,}600$ and it was found that

$$J_1 = 1.801,\ J_2 = 32.54,\ u = 5,\ C = 1764,\ A = 0.1542$$

The equation to be solved is therefore

$$f(u) = u^5 - u - 0.1542 \quad 0.$$

Substitution gives $f(0.9) = -0.155$, $f(1) = A = 0.154$. Hence the root sought is approximately $u_0 = 0.95$.

The following table gives the result of the successive approximations.

m	u_m	C
0	0.95	1.454×10^4
1	0.9554	1.459×10^4
2	0.9565	1.460×10^4
3	0.9570	1.460×10^4
4	0.9570	

The computations were made with four place logarithmic tables. It is seen that the convergence of the process is completed with $u_4 = u_3$ but that u_1 approximately and u_2 exactly reproduces the assumed value of C_2.

Though the precision of experimental data would not justify the use of seven place tables, it was deemed desirable to test the *method* by a computation with such tables. The convergence of the u's (to seven places) was complete at $u_9 = u_8$, but u_6 gave $C_2 = 1.459999 \times 10^4$, and u_7, u_8, u_9, gave $C_2 = 1.460000 \times 10^4$.

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