ELECTROMAGNETIC INDUCTION IN A HOMOGENEOUS SOLID CONDUCTING SPHERE ROTATING ABOUT AN AXIS PERPENDICULAR TO A UNIFORM ALTERNATING MAGNETIC FIELD

By Alexander Marcus

Abstract

(1) *Mathematical theory.* The differential equations of the field are solved by a method first given by H. Lamb. The induced currents are found to flow in concentric shells; to an observer moving with the sphere they would be equivalent to two sets of steady currents revolving in the sphere with frequencies equal respectively to the sum and the difference of the frequencies of the impressed alternating field and of rotation of the sphere. The currents may be supposed to diffuse from the surface inward in the usual manner; at higher frequencies the penetration is smaller, resulting in a "skin effect" for high frequencies. Equations are also derived for the resulting magnetic field and for the torque on the sphere and for the logarithmic decrement in the case of an oscillating conducting sphere. (2) To test the theoretical results, *measurements of the logarithmic decrement of a solid metal ball* oscillating about a vertical axis in an alternating field produced by a long horizontal solenoid were made, and the results for three balls were found to agree with values computed from the theoretical expression for the torque within the limit of experimental error, which was not greater than a few tenths per cent.

THE subject of electromagnetic induction in spherical conductors has been investigated by many writers and the special cases that have been treated thus far may be divided into two classes, dealing either with a sphere at rest in an alternating magnetic field or with a sphere rotating in a steady field. Solutions of these cases have been published by C. Niven,[1] H. Lamb,[2] J. Larmor,[3] J. J. Thomson,[4] H. Hertz,[5] and R. Gans.[6]

So far as the writer is aware, no one has hitherto published the solution of the problem of a sphere rotating in an alternating magnetic field. The general problem of rotation in an oblique field breaks up into two cases, for a magnetic field that is unsymmetrical to the axis of rotation may be resolved into two components, one that is symmetrical to the axis and another that is perpendicular to the axis. The alternating symmetrical component produces the same effect, so far as the currents in

[1] C. Niven, Phil. Trans. part II, 1881
[2] H. Lamb, Phil. Trans. part II, 1883
[3] J. Larmor, Phil. Mag. Jan. 1884
[4] J. J. Thomson, Recent Researches, pp. 546-557
[5] H. Hertz, Collected Papers
[6] R. Gans, Zeit. f. Math. und Physik, **48**, No. 1, 1902

the sphere are concerned, whether the sphere be stationary or rotating, for the number of lines of magnetic force that thread through any closed curve in the sphere remains unchanged by the rotation. We shall therefore proceed directly to the case of rotation in the magnetic field that is perpendicular to the axis.

Let us suppose that the sphere is rotating with constant angular velocity ω about the Z axis in an alternating field $H \cos pt$ parallel to the X axis. In this case the induced currents and the magnetic field are exactly the same as they would be in a stationary sphere placed in an alternating field revolving with angular velocity $-\omega$, referred to a set of axes X, Y, Z fixed relative to the apparatus producing the field. The components of the latter are derivable from the potential $Hx \cos pt$. As we shall use spherical coordinates throughout this paper we may write this expression as $Hr \sin \theta \cos \varphi \cos pt$. As viewed by an observer moving with the sphere, the impressed field is given by the gradient of $Hr \sin \theta \cos (\varphi_0 + \omega t) \cos pt$, since the only coordinate that is influenced by the motion is φ and the relation between φ and φ_0 is given by $\varphi = \varphi_0 + \omega t$ where φ_0 pertains to the system of axes revolving with the sphere.

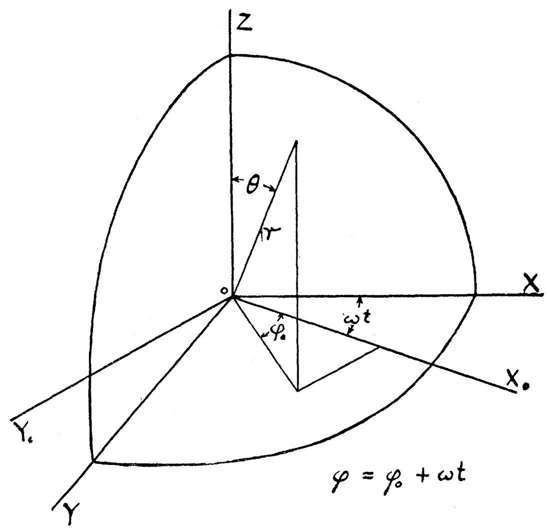

Fig. 1

Mathematically the problem is now reduced to the case of a stationary sphere in an alternating field whose components with respect to the r, θ and φ_0 coordinates are $[H \sin \theta \cos pt \cos (\varphi_0 + \omega t)]_r$, $[H \cos \theta \cos pt \cos (\varphi_0 + \omega t)]_\theta$, and $-[H \cos pt \sin (\varphi_0 + \omega t)]_\varphi$ or $\{\frac{1}{2}H \sin \theta \cos [\varphi_0 + (\omega + p)t] + \frac{1}{2}H \sin \theta \cos [\varphi_0 + (\omega - p)t]\}_r$ etc. To find the induced effects we may now employ Maxwell's equations for stationary media. The following symbols will be used in the equations: M, the resultant magnetic intensivity: E, the resultant electric intensity; r, θ, φ_0 the polar coordinates of a point; c, the velocity of light; λ, the conductivity of the metal; a, the radius of the sphere; n, the frequency of the currents in the primary coil; m, the frequency of rotation of the sphere. M and E are both measured in electromagnetic units.

The equations pertaining to a particle within the sphere are

$$\text{curl } M = 4\pi\lambda E; \tag{1}$$

$$\text{curl } E = dM/dt; \tag{2}$$

$$\text{div } E = 0 = \text{div} M; \tag{3}$$

$$\nabla^2 M = 4\pi\lambda dM/dt; \tag{4}$$

$$\nabla^2 E = 4\pi\lambda dE/dt. \tag{5}$$

As we may neglect the displacement currents in this case, we get for the region outside of the conductor the equations $\nabla^2 M = 0$ and $\nabla^2 E = 0$.

The impressed field gives rise to two types of electromagnetic disturbances, the free or transient vibrations and the forced vibrations. If we allow sufficient time for the free disturbance to subside, we need concern ourselves only with the forced vibrations. Moreover the problem of determining the precise mode in which the transient effects are dissipated in time and space in a stationary sphere has been completely solved by H. Lamb.[7] His solution will apply directly to the case of a revolving sphere, for as regards the transient effects the frequency of the impressed field is of no importance and as we have replaced the motion of the sphere by an appropriate variation of the impressed field, angular velocity of the sphere also can not influence the transients.

The part of the problem which has not been solved heretofore is based on the solution of the equation of the type $\nabla^2 M = 4\pi\lambda dM/dt$ which will also satisfy $\text{div} M = 0$ as well as the boundary conditions. We have already proved that relative to the rotor, the impressed field has two frequencies proportional to $p+\omega$ and $\omega - p$ respectively, where $p = 2\pi n$ and $\omega = 2\pi m$. We may therefore assume that M and E vary as ϵ^{qt} where $q = (\omega \pm p)j$ where $j = \sqrt{-1}$ and $k^2 = 4\pi\lambda q$. Eq. (4) may now be put into the form:[8]

$$\text{(I)} \quad \frac{\partial^2 (r^2 M_r)}{r^2 \partial r^2} + \frac{\partial(\sin\theta\, \partial M_r / \partial\theta)}{r^2 \sin\theta\, \partial\theta} + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2 M_r}{\partial\varphi^2} + k^2 M_r = 0,$$

with two similar expressions in M_θ and M_φ, and div $M = 0$ may be expanded into

$$\text{(II)} \qquad \frac{1}{r^2}\frac{\partial}{\partial r} r^2 M_r + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta} \sin\theta\, M_\theta + \frac{1}{r\sin\theta}\frac{\partial M_\varphi}{\partial\varphi} = 0.$$

[7] Lamb, Phil. Trans., part II, 1883

[8] H. Weber, Die Part. Diff. Gleich., pp. 336-337.

Lamb has long ago given two particular solutions of the systems (I) and (II) but only one of these solutions is appropriate to our problem. Bromwich[9] has recently proved that the general solution of Maxwell's equations can be derived by superposing two special solutions which can be obtained by taking first $E_r=0$ and secondly $M_r=0$. These two solutions are of a reciprocal character and when $E_r=0$ none of the components of M can vanish, while if $M_r=0$ none of the components of E can vanish. It is obvious that the particular solution which makes $M_r=0$ is inapplicable to this case.

The appropriate form of the solution is

$$\left.\begin{aligned} M_r &= -\sum_1^2 6f_1(k_nr)\frac{S_n}{r}\epsilon^{q_nt}; \\ M_\theta &= \sum_1^2 \left\{2f_0(k_nr)-k^2r^2f_2(k_nr)\right\}\frac{1}{r}\ \frac{\partial S_n}{\partial\theta}\epsilon^{q_nt}; \\ M_\varphi &= \sum_1^2 \left\{2f_0(k_nr)-k^2r^2f_2(k_nr)\right\}\frac{1}{r\sin\theta}\ \frac{\partial S_n}{\partial\varphi_0}\epsilon^{q_nt}; \end{aligned}\right\} \tag{6}$$

$$\left.\begin{aligned} E_r &= 0; \\ E_\theta &= -\sum_1^2 \frac{3q}{\sin\theta}f_1(k_nr)\frac{\partial S_n}{\partial\varphi_0}\epsilon^{q_nt}; \\ E_\varphi &= \sum_1^2 3qf_1(k_nr)\ \frac{\partial S_n}{\partial\theta}\epsilon^{q_nt}. \end{aligned}\right\} \tag{7}$$

$S_n=A_nr\sin\theta\ \epsilon^{j\varphi_0}$, where A is a constant. $f_0(k_nr)$, $f_1(k_nr)$, and $f_2(k_nr)$ all satisfy

$$\frac{d^2f_n}{dx^2}+\frac{2(n+1)}{x}\ \frac{df_n}{dx}+f_n=0,$$

where

$$x=k_nr,\ k_1=\tfrac{1}{2}\sqrt{2}\,(1+j)\ \sqrt{4\pi\lambda(\omega+p)},\text{ and } k_2=\tfrac{1}{2}\sqrt{2}\,(1+j)\ \sqrt{4\pi\lambda(\omega-p)}$$

and

$$q_1=j(\omega+p)\text{ and } q_2=j(\omega-p).$$

Eqs. (6) and (7) were obtained from the corresponding expressions in Cartesian coordinates given by Lamb in his third edition of "Hydrodynamics" on pages 572-573. The differential equation in f_n corresponds to Lamb's differential equation in R_n on page 478 of the Hydrodynamics. It is necessary to use the following recurrence relations between f_0, f_1 and f_2 in order to make (6) and (7) satisfy the field equations

$$f_{n+1}(x)=-\frac{1}{x}\ \frac{df_n(x)}{dx},\ \text{ and } x\frac{df_n}{dx}+(2\pi+1)f_n=f_{n-1}.$$

[9] Bromwich, Phil. Mag. (July 1919)

Since $\nabla^2 M=0$ outside of the sphere, the magnetic intensity due to the currents is derivable from a potential. We may therefore write for this part of the field

$$\left.\begin{aligned} M_r' &= -\sum_1^2 a^3 \frac{\partial}{\partial r}\left(\frac{\chi_n}{r^3}\right) \epsilon^{q_n t}; \\ M_\theta' &= -\sum_1^2 a^3 \frac{\partial}{r\partial\theta}\left(\frac{\chi_n}{r}\right) \epsilon^{q_n t}; \\ M_\varphi' &= -\sum_1^2 a^3 \frac{\partial}{r\sin\theta\,\partial\varphi}\left(\frac{\chi_n}{r^3}\right) \epsilon^{q_n t}; \end{aligned}\right\} \tag{8}$$

where $\chi = B_n\, r \sin\theta\ \epsilon^{j\varphi_0}$.

In order to find the A_n's and B_n's in (6), (7), and (8) we apply the boundary conditions for M. The tangential components of M on both sides of the surface of the sphere are equal, and since the permeability of the metal is 1, we may also equate the radial components of M on both sides of the boundary. These two conditions give us a pair of simultaneous equations in the A_n's and B_n's corresponding to each of the two frequencies.

At $r=a$

$$-6f_1(k_1a)A_1 \sin\theta\, \epsilon^{j\varphi_0}\epsilon^{j(p+\omega)t} = 2B_1 \sin\theta\, \epsilon^{j\varphi_0}\epsilon^{j(p+\omega)t} + \tfrac{1}{2}H\sin\theta\ \epsilon^{j\varphi_0}\epsilon^{j(p+\omega)t},$$

and

$$[2f_0(k_1a) - k^2a^2 f_2(k_1a)]A_1 \cos\theta\, \epsilon^{j\varphi_0}\epsilon^{j(p+\omega)t} = -B_1 \cos\theta\ \epsilon^{j\varphi_0}\epsilon^{j(p+\omega)t} + \tfrac{1}{2}H\cos\theta\, \epsilon^{j\varphi_0}\epsilon^{j(p+\omega)t}$$

with two similar equations connecting A_2 with B_2. The terms in H on the right represent the impressed field. Therefore

(a) $6f_1(k_1a)A_1 + 2B_1 = -\tfrac{1}{2}H;$

(b) $[2f_0(k_1a) - k^2a^2f_2(k_1a)]A_1 + B_1 = \tfrac{1}{2}H;$

but $3f_1(k_1a) = -f_0(k_1a) - k_1^2a^2f_2(k_1a)$, and substituting for $k_1^2a^2f_2(k_1a)$ in (b) we get

(c) $3[f_0(k_1a) + f_1(k_1a)]A_1 + B_1 = \tfrac{1}{2}H.$

From (a) and (b) we find

$$A_1 = \frac{H}{4f_0(k_1a)}, \quad \text{and} \quad B_1 = \frac{-H[3f_1(k_1a) + f_0(k_1a)]}{4f_0(k_1a)}.$$

Similarly we find

$$A_2 = \frac{H}{4f_0(k_2a)}, \quad \text{and} \quad B_2 = \frac{-H[3f_1(k_2a) + f_0(k_2a)]}{4f_0(k_2a)}.$$

The value of M due to the currents and the value of E are now given by the real components of the following expressions for space inside:

$$\left.\begin{aligned}
M_r &= -\frac{3}{2}H\sin\theta\left\{\frac{f_1(k_1r)}{f_0(k_1a)}\epsilon^{j[\varphi_0+(p+\omega)t]}+\frac{f_1(k_2r)}{f_0(k_2a)}\epsilon^{j[\varphi_0+(\omega-p)t]}\right\}\\
M_\theta &= \frac{3}{4}H\cos\theta\left\{\frac{f_0(k_1r)+f_1(k_1r)}{f_0(k_1a)}\epsilon^{j[\varphi_0+(p+\omega)t]}+\frac{f_0(k_2r)+f_1(k_2r)}{f_0(k_2a)}\epsilon^{j[\varphi_0+]\omega-p[t]}\right\}\\
M_\varphi &= -\frac{3}{4}jH\left\{\frac{f_0(k_1r)+f_1(k_1r)}{f_0(k_1a)}\epsilon^{j[\varphi_0+(p+\omega)t]}+\frac{f_0(k_2r)+f_1(k_2r)}{f_0(k_2a)}\epsilon^{j[\varphi_0+(\omega-p)t]}\right\}.
\end{aligned}\right\} \quad (9)$$

$$\left.\begin{aligned}
E_\varphi &= -j\frac{3}{4}Hr\cos\theta\left\{(\omega+p)\frac{f_1(k_1r)}{f_0(k_1a)}\epsilon^{j[\varphi_0+(p+\omega)t]}+(\omega-p)\frac{f_1(k_2r)}{f_0(k_2a)}\epsilon^{j[\varphi_0+(\omega-p)t]}\right\}\\
E_\theta &= \frac{3}{4}Hr\left\{(\omega+p)\frac{f_1(k_1r)}{f_0(k_1a)}\epsilon^{j[\varphi_0+(p+\omega)t]}+(\omega-p)\frac{f_1(k_2r)}{f_0(k_2a)}\epsilon^{j[\varphi_0+(\omega-p)t]}\right\}.
\end{aligned}\right\} \quad (10)$$

$f_0(x)$ may have any of the forms $\sin x/x$, $\cos x/x$, ϵ^{jx}/x, and $f_1(x)$ may have any of the forms

$$-\frac{1}{x}\frac{d}{dx}\frac{\sin x}{x},\quad -\frac{1}{x}\frac{d}{dx}\frac{\cos x}{x},\quad -\frac{1}{x}\frac{d}{dx}\frac{\epsilon^{jx}}{x}$$

The only form of $f_1(x)$ which is applicable to space inside, where x may become 0, is the first, and at $x=0$, $f_1(x)=1/3$.

The two cases of interest are first the one in which p and ω are small and ka small, and second, the one in which p and ω are large and ka large.

For small values of x, $f_0(x)=1$ (approx.), $f_1(x)=1/3$.

For large values of x, $f_0(x)=\epsilon^{jx}/2jx$, $\quad f_1(x)=\epsilon^{jx}/2x^2$.

Substituting in the general formulas (9) and (10), we get for kr small, inside:

$$E_\theta=\tfrac{1}{4}Hr\{(\omega+p)\cos[\varphi_0+(\omega+p)t]+(\omega-p)\cos[\varphi_0+(\omega-p)t]\} \quad (11)$$

etc.

$$M_r=-\tfrac{1}{2}H\sin\theta\{\cos[\varphi_0+(\omega+p)t]+\cos[\varphi_0+(\omega-p)t]\} \quad (12)$$

etc. For large values of kr inside, we must take the real parts of the following expressions:

$$E_\theta=\frac{3}{8\sqrt{\pi}}\frac{a}{r}\left\{\sqrt{\frac{p+\omega}{\lambda}}\,\epsilon^{-m_1(a-r)-j[m_1(a-r)+\pi/4]}\,\epsilon^{j[\varphi_0+(p+\omega t]}\right.$$

$$\left.+\sqrt{\frac{\omega-p}{\lambda}}\,\epsilon^{-m_2(a-r)-j[m_2(a-r)+\pi/4]}\,\epsilon^{j[\varphi_0+(\omega-p)t]}\right\} \quad (13)$$

$$M_r=-\frac{3H}{4\sqrt{\pi}}\,\frac{a}{r^2}\sin\theta\left\{\frac{\epsilon^{-m_1(a-r)-jm_1(a-r)}}{\sqrt{\lambda(\omega+p)}}\cos[\varphi_0+(\omega+p)t]\right.$$
$$\left.+\frac{\epsilon^{-m_2(a-r)-jm_2(a-r)}}{\sqrt{\lambda(\omega-p)}}\cos\,[\varphi_0+(\omega-p)t]\right\}\quad(14)$$

$m_1=\sqrt{2\pi\lambda(\omega+p)}$, $m_2=\sqrt{2\pi\lambda(\omega-p)}$ or $\sqrt{2\pi\lambda(p-\omega)}$ according as $p<\omega$, or $p>\omega$.

From the expressions for M and E it follows that an observer moving with the sphere will detect a pair of revolving fields and two sets of revolving currents.

Let us now consider the penetration of the disturbance within the solid. For small values of p and ω the current density on any shell varies directly as the radius of the shell, being a maximum at the surface and zero at the center. The impressed magnetic field being uniform, the same inductive effect is produced on all concentric shells.

For p and ω large, the penetration within the conductor takes place as a propagation of two highly damped waves inward from the surface. At the surface, the current density increases with the conductivity of the material, for the components of the current density λE vary as $\sqrt{\lambda(\omega\pm p)}$. At any depth $a-r$ beneath the surface, the current density decreases with an increase in λ owing to the very rapid decrease in $\epsilon^{-\sqrt{2\pi\lambda(\omega\pm p)}}$. The surface of a good conductor therefore screens the interior against rapidly varying external electromagnetic disturbances irrespective of whether the induced effects be due to the rotation of the conductor or the variation of the field or to the two combined. In the case of a conductor rotating in a field that is not symmetrical to the axis of rotation the surface can shield the interior only against non-symmetrical components, for the rotation in the symmetrical field is not accompanied by induced currents. These facts lead to the view that the direct induction creates only superficial currents at the start. These currents, were there no resistance, would continue to flow undiminished and would prevent the interior of the solid from being affected. The sphere may be regarded as made up of an infinite number of conducting shells. As the currents of the outer shell decay by resistance this shell ceases to be a complete screen. Currents are then created in the inner shells by direct induction exactly as in the outer one, and thus the whole sphere becomes pervaded.

The Mechanical Reaction Between the Currents in the Sphere and the Impressed Magnetic Field

If F be the mechanical force per unit volume, then (Maxwell's Treatise, vol. II, §603), $F=[c\times M]$, the right hand member being the vector product of the impressed magnetic field alone (since the rotor does not produce a torque on itself), and the current at the point in question. Let X, Y, Z, a, b, c, u, v, w be the components of F, M and C along the x_0, y_0 and z axes in the moving system. Then

$$F=\begin{vmatrix} i & j & k \\ u & v & w \\ a & b & c \end{vmatrix}$$

and the couple per unit volume is equal to

$$[r\times F]=\begin{vmatrix} i & j & k \\ x_0 & y_0 & z \\ X & Y & Z \end{vmatrix}$$

where $r=x_0i+y_0j+zk$. The couple around the axis of z is $\iiint(x_0Y-y_0X)\,dx\,dy\,dz$; but $X=bw-cv$ and $Y=cu-aw$. Substituting we get, $y_0X-x_0Y=by_0w-cy_0v-cx_0u+ax_0w=(ax_0+by_0+cz)\,\omega-c\,(ux_0+vy_0+wz)$. Since $E_r=0$, $wz+vy_0+ux_0=0$, also $ax_0\times by_0+cz=rM_r$ and $\omega=\lambda E_\theta\sin\theta$; hence the torque around the z axis is $\iiint\lambda E_\theta M_r r\sin\theta\,dS$ where $dS=r^2\sin\theta\,d\theta\,d\varphi\,dr$ or

$$\text{Torque}=\int_{r=0}^{r=a}\int_{\theta=0}^{\theta=\pi}\int_{\varphi=0}^{\varphi=2\pi}\lambda E\theta\,M_r\,r^3\sin^2\theta\,d\theta\,d\varphi\,dr. \tag{15}$$

Inserting for M_r only the external field, we find for the average resultant torque

$$\mathcal{G}=\pi\lambda H^2\left\{(\omega+p)\int_0^a\frac{f_1(k_1r)}{f_0(k_1a)}r^4dr+(\omega-p)\int_0^a\frac{f_1(k_2r)}{f_0(k_2a)}r^4dr\right\} \tag{16}$$

For p and ω small,

$$\frac{f_1(k_1r)}{f_0(k_1a)}=\frac{f_1(k_2r)}{f_0(k_2a)}=\frac{1}{3},$$

hence the resultant torque is

$$\mathcal{T}=\tfrac{2}{15}\pi a^5\lambda H^2\omega. \tag{17}$$

In so far as the reaction upon the impressed field is concerned, the currents in the sphere flow as though they were confined to tubes coinciding with great circles through the axis of rotation, for in the expression for the torque the only component of the current that appears is E_θ. Moreover if we calculate the line integral of $E\varphi$ about a line whose plane is perpendicular to the axis of rotation we find that it vanishes, i.e.

$$\int_0^{2\pi}E\varphi a\sin\theta\,d\varphi_0=0.$$

This must necessarily be true since there is no impressed field parallel to the axis of rotation.

Experimental Confirmation

The foregoing theoretical development was tested experimentally in the following manner. A metal ball was suspended by means of a torsion fibre at the center of a long solenoid carrying a steady alternating current. The suspension fibre was perpendicular to the direction of the field. The ball was given a start and allowed to oscillate freely in air, and an observation of the decrement of the oscillation was taken. The field was then thrown on and the decrement was again determined. The difference between the two decrements gave the component due to the induced currents alone. The latter could be predetermined from theory as follows.

The equation of motion of the ball is given by

$$P\frac{d^2\varphi}{dt^2}+k\frac{d\varphi}{dt}+\tau\varphi=0 \tag{18}$$

where P is the moment of inertia and k the coefficient of damping including air damping as well as electromagnetic damping. τ is the restoring moment for unit angular deflection of the elastic suspension fibre. The solution of (18) is given by

$$\varphi=\varphi_0\epsilon^{-kt/2p}\cos(2\pi t/T) \tag{19}$$

where T is the period.

If δ be the logarithmic decrement of the oscillation we can prove by means of (19) that $\delta=kT/4P$.

In the experiments the period of oscillation of the ball ranged from about four to fifteen seconds; therefore it was permissible to compute the theoretical value of δ on the assumption that the angular velocity ω was constant at any time.

The first experimental test was performed to check the result given by (17) for the average resultant torque. This expression for $\mathcal{T}$ should be strictly true, according to the theory, for impressed fields of zero frequency. Accordingly, metal balls having different resistivities were suspended in turn in uniform constant magnetic fields and allowed to oscillate. The damping due to the induced currents was measured in each case and the logarithmic decrement for each ball was compared with the value given by the theoretical expression. The latter is found in the following manner. $\delta=kT/4P$ where $k=\mathcal{T}\omega$ and, for a sphere, $P=\frac{2}{5}a^2M$; hence $\delta=TH^2\pi a^3/12M\sigma$, where σ is the resistivity of the material or $1/\lambda$.

In every case the values of δ (the decrement) as given by theory and experiment differed by not more than one tenth of one per cent. This method was later employed to determine σ for copper and for lead from the values of δ obtained experimentally. The values of σ obtained from the formula agreed with the published values as closely as the different published values agreed among themselves.

The test of the theory was then extended to those cases for which the expression for δ could be put into the form of a rapidly converging power series in the variable $v=2\pi a\sqrt{f/\sigma}$ where a is the radius of the ball and f the impressed frequency. The reason for thus restricting the test was that the separation of the real and the imaginary parts of the integral of (16) becomes extremely laborious except when we restrict the investigation to cases for which $v<\pi$. In the latter cases the expansion is made as follows:

$$\int_0^a \frac{f_1(k_n r)}{f_0(k_n a)} r^4 dr = \frac{a^5}{x^2}\left[1+\frac{3\cot x}{x}-\frac{3}{x^2}\right]$$

where $x=k_n a$; but

$$\cot x=\frac{1}{x}-\frac{x}{3}-\frac{x^3}{45}-\frac{2x^5}{945}-\frac{x^7}{4725}.$$

On substituting in (16) and changing the variable x into v by the relation $v=2\pi a\sqrt{f/\sigma}$ we get

$$\mathcal{T}=\tfrac{2}{15}(H^2\pi a^5/\sigma)(1-0.1143v^4-7.73\times10^{-3}v^8\ldots)\omega,$$

and

$$\delta=(TH^2\pi a^3/12M\sigma)(1-0.1143v^4-7.73\times10^{-3}v^8\ldots).$$

Apparatus

The ball was suspended with its center in the middle of a solenoid 122.4 cm long having an average diameter of 7.1 cm. The solenoid consisted of two separate but identical parts each wound with four layers of No. 16 cotton-covered copper wire. The ball was suspended by means of a phosphor-bronze ribbon within a box mounted on leveling screws. When the solenoids were placed in contact over the ball the two end turns touched at nearly all points. There was a very narrow short slit at the top to allow the suspension to pass through. Readings of successive amplitudes were taken by observing the deflection of a beam of light on a scale about a meter away.

Balls of different non-magnetic metals were used. The frequencies of the impressed field ranged from zero to 580 cycles per second. In every case it was necessary to determine the resistivity of the sample of metal used. This was done by keeping the field constant and measuring the decrement. The resistivity was computed from $\delta_0=TH^2\pi a^3/12M\sigma$. This expression for δ_0 follows directly from our theoretical expression for δ_0 in an alternating field by putting $f=0$.

TABLE I

Comparison of experimental and theoretical values of the decrement ρ.

Substance	a	M	T	i (effective)	f	ρ (exp.)	ρ (theor.)
Copper	0.96 cm	33.76 g	14.9 sec.	1.66 amp.	64.7	1.135	1.133
Copper	0.53	5.86	4.62	4.00	580	1.013	1.013
Lead	1.17	79.86	5.24	3.17	62	1.003	1.003

The effective value of H in the expression for δ is given by $H=30.308i$ where i is in (effective) amperes.

In each case the experimental value of ρ is the mean of a large number of observations and the probable error of the mean was in no case greater than one part in a thousand. The maximum deviation of a single observation from the mean was about five parts in a thousand.

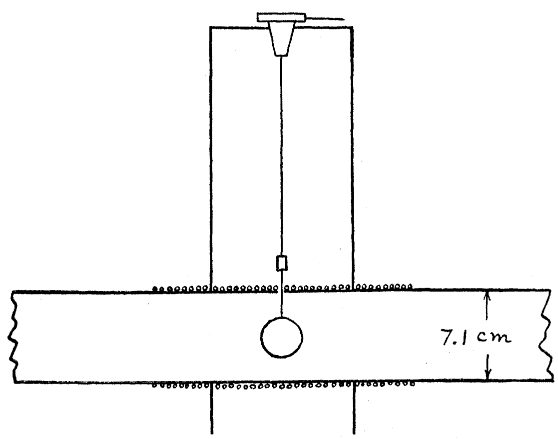

Fig. 2. Oscillating sphere suspended inside a solenoid, 122 cm long.

A comparison of the values of ρ as found experimentally with those computed from the theory shows that within the limits of experimental error, theory and experiment give the same values.

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