TRANSFORMATION OF ELECTRIC AND MAGNETIC FORCES IN A PLANE WAVE, IN A PLANE NORMAL TO THE DIRECTION OF RELATIVE MOTION OF TWO OBSERVERS

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Abstract

Two observers are assumed to move along the X axis and to measure the components of the electric intensity $E(E_1, E_2, E_3)$ and magnetic intensity $H(H_1, H_2, H_3)$ of a plane electromagnetic wave propagated at an arbitrary angle to the X axis. According to the theory of relativity, the two observers will find the same values of the components E_1 and H_1 along the X axis. The components E_2, E_3, H_2, H_3, along the Y and Z axes, are replotted in a novel manner in the so-called *computing plane* which consists of two sets of oblique co-ordinates. The angle comprised by each set is uniquely determined by the relative velocity of the two observers. In the computing plane a transformation point M is located such that its abscissa and ordinate with respect to one set of oblique axes gives the components E_2 and H_3 of the wave intensities as measured by one of the observers, while its co-ordinates with respect to the other set of axes gives the values of the same components as measured by the other observer. Another transformation point N gives the components E_3 and H_2 for the two observers. It is shown that the loci of M and N are ellipses. The results obtained are shown to check with the well-known equations of the theory of relativity. The location of the points M and N is shown in two simple special cases, and instructions are given for locating them in the general case.

CONSIDER an observer S, with an orthogonal right-handed system of axes XYZ which are stationary with respect to himself, and let a plane electromagnetic wave be propagated in some direction which does not coincide with any of the coordinate axes. Let a second observer S' be in uniform motion with respect to S, in the positive direction of the X axis, and let the observer S' refer his observations to an orthogonal right-handed system of axes $X'Y'Z'$ moving with him, and such that the X' axis coincides with the X axis while the Y' and Z' axes remain parallel to the corresponding axes Y and Z of the first observer. Furthermore, let the velocity of light in vacuum be taken as unity, and the relative velocity q of the two observers be expressed as a fraction of the velocity of light.

The S observer determines the components, parallel to the coordinate axes, of the electric and magnetic intensities (E and H) of the wave, and so does the S' observer. Let the S observer find the amplitude values of the components of the electric intensity parallel to the X, Y, and Z axes to be E_1, E_2, E_3 respectively and let the corresponding values of the magnetic intensity be H_1, H_2, H_3. Let the S' observer measure the values E_1', E_2', E_3', H_1', H_2', H_3'. According to the restricted theory of relativity, we have the following relations among these quantities:[1]

$$E_1' = E_1 \tag{1}$$
$$E_2' = \beta(E_2 - q\, H_3) \tag{2}$$
$$E_3' = \beta(E_3 + q\, H_2) \tag{3}$$

and also

$$H_1' = H_1 \tag{4}$$
$$H_2' = \beta(H_2 + q\, E_3) \tag{5}$$
$$H_3' = \beta(H_3 - q\, E_2) \tag{6}$$

where

$$\beta = (1 - q^2)^{-\frac{1}{2}} \tag{7}$$

The sinusoidal function which expresses variations of E' and H' with the time is here omitted, since we are concerned only with their amplitudes; the questions of aberration and Doppler effect are not considered in this article. The two observers find the same values of the electric and magnetic components in the direction of their relative motion, but different values of the Y and Z components.

The purpose of the present article is to show that Eqs. (2), (3), (5), and (6) admit of a simple graphical interpretation. The diagrams shown below not only permit to visualize these relationships and to check the signs of the individual components, but may lead to new relationships not apparent in the analytical equations.

In Fig. 1 two sets of *oblique axes* are shown, viz., YOZ and $Y'OZ'$. This figure and the axes are intended only for computation purposes, that is, for plotting the values of the electric and magnetic components, and have nothing to do with the actual relations in space. In other words, Fig. 1 represents an arbitrary plane for a graphical representation of the abovementioned equations, and we shall refer to it as the *computing* plane. The two sets of axes are obtained as follows. Let OC be an arbitrary reference axis, and OY_0 and OZ_0 two mutually perpendicular directions at 45° to OC. Then the YOZ set of axes is obtained by expanding Y_0OZ_0 symmetrically by an angle α, while the $Y'OZ'$ set is obtained by contracting the Y_0OZ_0 set by the same angle α. OC is the common bisector for all the three sets of axes. The acute angle α is determined by the relationship

$$\sin\alpha = q \tag{8}$$

[1] See, for example, O. W. Richardson, *The Electron Theory of Matter*, 1916, p. 309.

where q, as mentioned above, is the relative velocity of the two observers, expressed as a fraction of the velocity of light. Eq. (7) becomes

$$\beta = \sec \alpha \tag{9}$$

We shall now prove that *for any given plane electromagnetic wave and a given relative velocity q of the two observers, there are in the computing plane two transformation points, M and N, equivalent to the Eqs. (2), (3), (5), and*

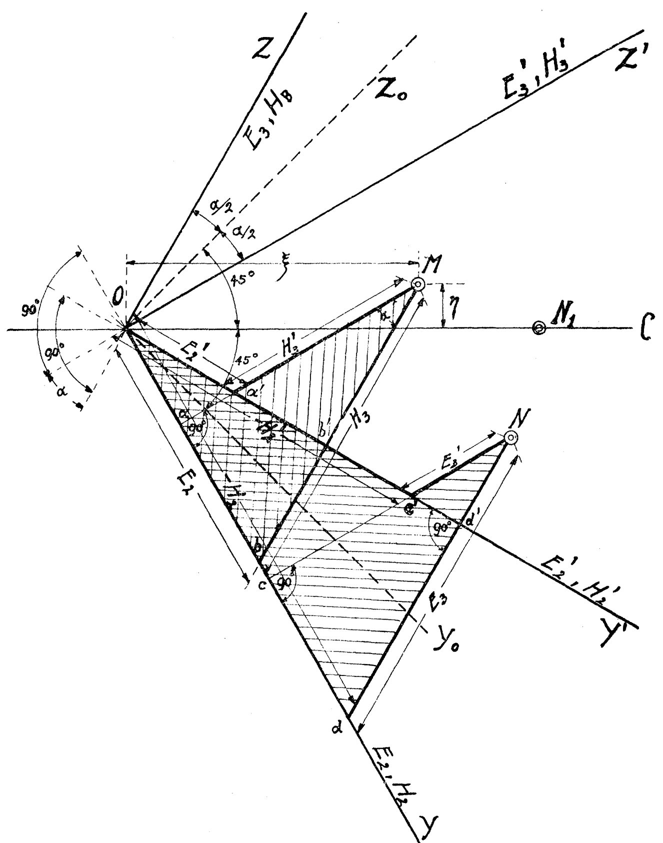

Fig. 1. The computing plane with transformation points M and N.

(*6*). To obtain point M, we plot along OY the value $Ob = E_2$, measured by the S observer, and draw the ordinate $bM = H_3$, parallel to the OZ axis. From the geometry of the figure, the abscissa Oa' of the same point M in the $Y'OZ'$ system is equal to the electric component E_2' measured by the S' observer, and the ordinate $a'M$ is equal to the magnetic component H_3' found by the same observer. Namely,

$$E_2' \cos \alpha = E_2 - H_3 \sin \alpha \tag{10}$$

Multiplying both sides of this equation by sec α, gives

$$E_2' = \sec\alpha\ (E_2 - H_3 \sin\alpha) \tag{11}$$

Substituting for sin α and sec α their values from Eqs. (8) and (9), Eq. (2) is obtained. Similarly, from Fig. 1,

$$H_3' \cos\alpha = H_3 - E_2 \sin\alpha \tag{12}$$

Multiplying both sides of this expression by sec α and using again Eqs. (8) and (9), an expression is obtained, identical with Eq. (6).

The transformation point N in the YOZ system has the abscissa equal to $-H_2$ and the ordinate E_3. The necessity for a minus sign before H_2 will be seen by considering an electromagnetic wave propagated along the X axis, with the electric intensity in the first YZ quadrant. Three of the projections, E_2, E_3, and H_3, are then positive, while H_2 is negative. Since in Fig. 1 both transformation points, M and N, are shown in the first quadrant, H_2 must be taken with the minus sign.

To prove that the transformation point N represents Eqs. (3) and (5), we write, from Fig. 1

$$E_3' \cos\alpha = E_3 + H_2 \sin\alpha \tag{13}$$

$$-H_2' \cos\alpha = -H_2 - E_3 \sin\alpha \tag{14}$$

Multiplying these equations by sec α, and using Eqs. (8) and (9), the foregoing expressions are converted into Eqs. (3) and (5).

We thus see that the relations (2), (3), (5), and (6) can be represented by the cross-hatched quadrilaterals $Oa'Mb$ and $Oc'Nd$. It is understood of course that in reality H_2 is normal to H_3 and E_2 is normal to E_2, and that the computing plane is to be used only for transforming the individual components from one observer to the other and not for the total values of E and H. There is nothing in Fig. 1 to indicate that one of the observers is "stationary" and the other "moving." The diagram merely states that if one of the observers measures certain values, say E_2 and H_3, then the observer whose *relative* velocity is q in the positive direction of the X axis, will find the values E_2' and H_3'.

It is of interest to find the locus of the transformation points M and N when the relative velocity q or (which is the same) the angle α varies. Since the position of the bisector OC remains the same, it is natural to orient M and N with respect to OC. Let the orthogonal co-ordinates of M be ξ and η. We then have

$$\xi = (H_3 + E_2) \cos (45° + 0.5\alpha) \tag{15}$$

$$\eta = (H_3 - E_2) \sin (45° + 0.5\alpha) \tag{16}$$

Eliminating α from these two equations gives an ellipse

$$\xi^2/(H_3+E_2)^2 + \eta^2/(H_3-E_2)^2 = 1 \tag{17}$$

A similar equation may be derived for the locus of the transformation point N.

Two Special Cases

(a) Let us consider the simplest possible case of a plane wave propagated along the X axis, with the electric intensity along the Z axis and the magnetic intensity along the negative Y axis. Let the magnitudes of both intensities be A. Then $E_1=E_2=0$; $H_1=H_3=0$; $E_3=A$; $H_2=-A$. In the computing plane (Fig. 1) the transformation point M coincides with the origin O, and the point N (denoted by N_1) lies on the bisector OC. It will be seen from the sketch that $E_2'=H_3'=0$ and

$$E_3' \cos(45°-0.5\alpha)=E_3 \cos(45°+0.5\alpha) \tag{18}$$

from which

$$E_3'=-H_2'=A \tan(45°-0.5\alpha)=A\sqrt{(1-q)/(1+q)} \tag{19}$$

This shows the apparent reduction in the magnitude of the two intensities for an observer traveling at a velocity q in the direction of propagation of the wave. The elliptic locus degenerates into the straight line ON'.

(b) The example considered by Richardson[1] is shown in the computing plane in Fig. 2. The direction of the propagation of the wave is in the XY plane, at an angle θ to the X-axis. The angle θ is considered positive towards the negative Y axis. The electric intensity is in the direction of

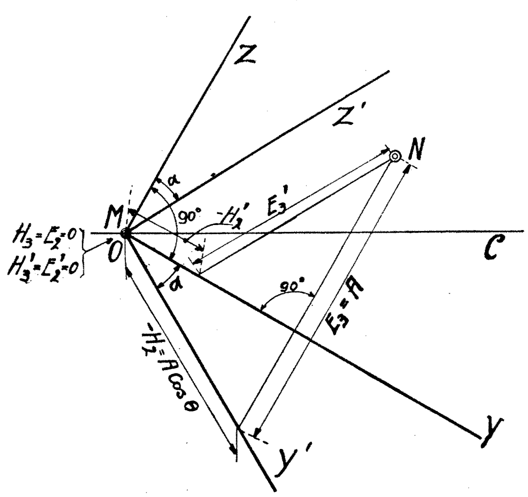

Fig. 2. Transformation of the E and H components of a wave propagated at an angle θ to the X axis, E being directed along the Z axis.

the Z axis. Here we have $E_1=E_2=0$; $E_3=A$; $H_1=-A\sin\theta$; $H_2=-A\cos\theta$; $H_3=0$. The transformation point M again coincides with the origin; the transformation point N lies above the bisector OC. We see from the diagram that $H_3'=E_2'=0$; the quantities E_3' and H_2' will also be found to agree with the expressions given by Richardson.

THE CO-ORDINATES OF M AND N IN THE GENERAL CASE

Let l, m, n be the direction cosines of the line of propagation of the wave, let its electric intensity be defined by its direction cosines l_e, m_e, n_e, and let the magnetic force H be similarly determined by the direction cosines l_h, m_h, n_h. Since the direction of propagation and the directions of E and H form three mutually perpendicular lines, we have:[2]

$$\begin{aligned} ll_e+mm_e+nn_e&=0\\ ll_h+mm_h+nn_h&=0\\ l_el_h+m_em_h+n_en_h&=0 \end{aligned} \tag{20}$$

and also the usual parallelopiped conditions

$$\begin{aligned} l^2+m^2+n^2&=1\\ l_e^2+m_e^2+n_e^2&=1\\ l_h^2+m_h^2+n_h^2&=1 \end{aligned} \tag{21}$$

A wave is completely determined by four out of the nine direction cosines and by its intensity $E=H=A$. The other direction cosines can be determined from Eqs. (20) and (21). Thus, for a given wave we may write

$$E_1=Al_e;\quad H_1=Al_h; \tag{22}$$

$$E_2=Am_e;\ H_3=An_h; \tag{23}$$

$$E_3=An_e;\ H_2=Am_h. \tag{24}$$

The values given by Eqs. (22) are the same for the S and S' observers. The values given by Eqs. (23) determine the transformation point M (Fig. 1) and those given by Eqs. (24) determine the transformation point N. It must be remembered that the abscissa of the point N is equal to H_2 with the minus sign. Having located M and N, we draw Ma' and Nc' parallel to Z'. This will give the values of the electric and magnetic intensities for the S' observer.

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[2] See, for example, Snyder and Sisam, Analytic Geometry of Space, Chap. III.