# EXTENSION OF MAXWELL'S SERIES FORMULA FOR THE MUTUAL INDUCTANCE OF COAXIAL CIRCLES.

By J. G. Coffin.

THE most important formula for the mutual inductance of coaxial circles is the exact formula in elliptic integrals given by Maxwell:[1]

$$M = 4\pi\sqrt{Aa}\left\{\left(\frac{2}{k} - k\right)F - \frac{2}{k}E\right\}, \qquad (1)$$

in which $A$ and $a$ are the radii of the two circles, $d$ is the distance between their centers, and

$$k = \frac{2\sqrt{Aa}}{\sqrt{(A+a)^2 + d^2}} = \sin\gamma.$$

$F$ and $E$ are the complete elliptic integrals of the first and second kind, respectively, to modulus $k$.

He also obtained an expression for the mutual inductance between coaxial circles in the form of a converging series which is sometimes more convenient to use, but has the especial importance of being integrable over the cross sections of coaxial coils thus giving rise to many valuable formulæ. It is thus seen to be a formula of fundamental importance. Maxwell however carried the series to terms of the third degree in $c/a$ and $d/a$ only. $A$ and $a$ are the radii of the two circles, $c = A - a$, $d$ is their distance apart. His formula includes only the terms of the third degree of the following one.

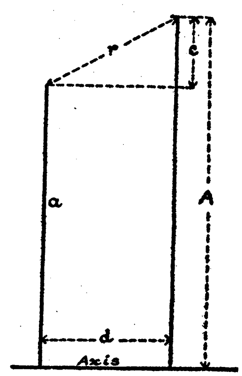


Fig. 1.

$$M = 4\pi a\left\{\log\frac{8a}{r}\left(1 + \frac{c}{2a} + \frac{c^2+3d^2}{16a^2} - \frac{c^3+3cd^2}{32a^3} + \frac{17c^4+42c^2d^2-15d^2}{1{,}024a^4}\right.\right.$$
$$\left. - \frac{19c^5+30c^3d^2-45cd^4}{2{,}048a^5} + \cdots\right) - \left(2 + \frac{c}{2a} - \frac{3c^2-d^2}{16a^2} + \frac{c^3-6cd^2}{48a^3}\right. \qquad (2)$$
$$\left.\left. + \frac{19c^4+534c^2d^2-93d^4}{48\cdot 128a^4} - \frac{379c^5+3{,}030c^3d^3-1{,}845cd^4}{480\cdot 128a^5} + \cdots\right)\right\}.$$

[1] Electricity and Magnetism, 4, p. 301, 1907.

E. B. Rosa and Louis Cohen[1] extended Maxwell's formula to terms of the fifth degree by expanding the terms of formula (1) in series and substituting them in it. Their result is formula (2). Nagaoka[2] confirmed this formula including terms of the fourth degree by means of an expansion of a formula due to him in the $q$-series of Jacobi.

Coffin[3] in order to deduce a formula for the self-inductance of a cylindrical coil deduced a formula equivalent to Maxwell's in which $c = 0$, that is, for equal circles, up to and including terms of the eighth degree, which gives with considerable exactness the mutual inductance of equal circles up to $d = a$. Coffin's formula is

$$M=4\pi a\left\{\log\frac{8a}{d}\left(1+\frac{3d^2}{16a^2}-\frac{15d^4}{8\times 128a^4}+\frac{35d^6}{128^2a^6}-\frac{1{,}575d^8}{2\times 128^3a^8}+\cdots\right)\right.$$
$$\left.-\left(2+\frac{d^2}{16a^2}-\frac{31d^2}{16\times 128a^4}+\frac{247d^6}{6\times 128^2a^6}-\frac{7{,}795d^8}{8\times 128^3a^8}+\cdots\right)\right\}. \tag{3}$$

This formula has been amply verified in the form it is and in the form of an expression for the self-inductance of a cylindrical current sheet deduced from it, by many numerical calculations compared with numerical results deduced from absolute formulæ. In 1908 Havelock[4] deduced an expression for the mutual inductance of coaxial circles depending on certain definite integrals of Bessel functions which he expanded in series. His formula is

$$M = 4\pi\sqrt{Aa}\left\{\log\frac{8\sqrt{Aa}}{r}\left[1+\frac{3}{16}\left(\frac{r^2}{Aa}\right)-\frac{15}{1{,}024}\left(\frac{r^2}{Aa}\right)^2+\cdots\right]\right.$$
$$\left.-\left[2+\frac{1}{16}\left(\frac{r^2}{Aa}\right)-\frac{31}{2{,}048}\left(\frac{r^2}{Aa}\right)^2+\cdots\right]\right\}. \tag{4}$$

This formula is simpler for use than Maxwell's but it is not in a form suitable for integration. Rosa noticed however that when $A = a$ Havelock's formula reduces to Coffin's formula (3) and hence it may be extended as follows:

$$M=4\pi\sqrt{Aa}\left\{\left[1+\frac{3}{16}\alpha-\frac{15}{1{,}024}\alpha^2+\frac{35}{128^2}\alpha^3-\frac{1{,}575}{2\times 128^3}\alpha^4+\cdots\right]\log\frac{8\sqrt{Aa}}{r}\right.$$
$$\left.-\left[2+\frac{1}{16}\alpha-\frac{31}{2{,}048}\alpha^2+\frac{247}{6\times 128^2}\alpha^3-\frac{7{,}795}{8\times 128^3}\alpha^4+\cdots\right]\right\}, \tag{5}$$

where $r^2 = c^2 + d^2$, $\alpha = r^2/Aa$.

[1] Bulletin of the Bureau of Standards, 2, p. 364, 1906.
[2] Nagaoka, Phil. Mag., 6, p. 19, 1903.
[3] J. G. Coffin, Bulletin of the Bureau of Standards, 2, p. 113, 1906.
[4] Havelock, Phil. Mag., 15, p. 332, 1908.

It occurred to the writer that Rosa's extension of Maxwell's formula (2) might be confirmed and new terms deduced in the following manner. This will be especially valuable, as the confirmation will be by a method independent of both Maxwell's and Rosa's.

Since $A = a + c$

$$\alpha = \frac{r^2}{Aa} = \frac{d^2 + c^2}{(a + c)a} = \frac{d^2 + c^2}{a^2}\frac{1}{\frac{a + c}{a}} = (\eta^2 + \epsilon^2)(1 + \epsilon)^{-1},$$

$$Aa = (a + c)a = a^2(1 + \epsilon),$$

$$\sqrt{Aa} = a(1 + \epsilon)^{\frac{1}{2}},$$

$$\log \frac{8\sqrt{Aa}}{r} = \log \frac{8a(1 + \epsilon)^{\frac{1}{2}}}{r} = \log \frac{8a}{r} + \log (1 + \epsilon)^{\frac{1}{2}},$$

where $c/a = \epsilon$ and $d/a = \eta$.

With these substitutions Havelock's formula (5) becomes

$$\begin{aligned} M = 4\pi a \Bigg\{ & \log \frac{8a}{r} \Bigg( (1+\epsilon)^{\frac{1}{2}} + \frac{3}{16}(\eta^2+\epsilon^2)(1+\epsilon)^{-\frac{1}{2}} \\ & \qquad - \frac{15}{1{,}024}(\eta^2+\epsilon^2)^2(1+\epsilon)^{-\frac{3}{2}} + \text{etc.} \Bigg) \\ & + \tfrac{1}{2}\log(1+\epsilon)\Bigg( (1+\epsilon)^{\frac{1}{2}} + \frac{3}{16}(\eta^2+\epsilon^2)(1+\epsilon)^{-\frac{1}{2}} \\ & \qquad - \frac{15}{1{,}024}(\eta^2+\epsilon^2)^2(1+\epsilon)^{-\frac{3}{2}} + \text{etc.} \Bigg) \\ & - \Bigg( 2(1+\epsilon)^{\frac{1}{2}} + \frac{1}{16}(\eta^2+\epsilon^2)(1+\epsilon)^{-\frac{1}{2}} \\ & \qquad - \frac{31}{16\times 128}(\eta^2+\epsilon^2)^2(1+\epsilon)^{-\frac{3}{2}} + \text{etc.} \Bigg). \end{aligned} \tag{6}$$

By expanding the parentheses $(1 + \epsilon)^n$ by means of the binomial theorem and using the formula for the $\log (1 + \epsilon) = \epsilon - \frac{1}{2}\epsilon^2 + \frac{1}{3}\epsilon^3 - \frac{1}{4}\epsilon^4 + \cdots$, all of which are convergent for $\epsilon = c/a < 1$ or for $c < a$ we have obtained an extension of Maxwell's formula (2) up to and including terms of the eighth degree.

This extension confirms the coefficients of Rosa and Cohen's extension and may be expected to give formulæ on integration over large cross sections, which converge rapidly and are quite exact for values of $c$ or of $d$ almost as great as the radius $a$. This formula is as follows

$$
\begin{aligned}
M = 4\pi a \Bigg\{ \log \frac{8a}{r} \Bigg( & 1 + \frac{c}{2a} + \frac{c^2+3d^2}{16a^2} - \frac{c^3+3cd^2}{32a^3} + \frac{17c^4+42c^2d^2-15d^4}{1,024a^4} \\
& - \frac{19a^5 + 30c^3d^2 - 45cd^4}{16 \cdot 128a^5} \\
A \quad & + \frac{89c^6 + 45c^4d^2 - 345c^2d^4 + 35d^6}{128^2a^6} \\
B \quad & - \frac{109c^7 - 63c^5d^2 - 525c^3d^4 + 175cd^6}{2 \cdot 128^2a^7} \\
C \quad & + \frac{8,921c^8 - 13,692c^6d^2 - 43,050c^4d^4 + 32,900c^2d^6 - 1,575d^8}{2 \cdot 128^3a^8} \Bigg) \\
& - \Bigg( 2 + \frac{c}{2a} - \frac{3c^2-d^2}{16a^2} + \frac{c^3-6cd^2}{48a^3} + \frac{19c^4+534c^2d^2-93d^4}{48 \cdot 128a^4} \\
& - \frac{379c^5 + 3,030c^3d^2 - 1,845cd^4}{480 \cdot 128a^5} \\
A' \quad & + \frac{2,629c^6 + 12,045c^4d^2 - 17,445c^2d^4 + 1,235d^6}{30 \cdot 128^2a^6} \\
B' \quad & - \frac{27,833c^7 + 71,169c^5d^2 - 225,225c^3d^4 + 50,575cd^6}{420 \cdot 128^2a^7} \\
C' \quad & + \frac{5,204,309c^8 + 5,304,852c^6d^2 - 45,499,650c^4d^4 + 21,735,700c^2d^6 - 818,475d^8}{840 \cdot 128^3a^8} \Bigg) \Bigg\}.
\end{aligned}
\tag{7}
$$

The terms marked $A$, $B$, $C$, $A'$, $B'$ and $C'$ are the newly derived ones. Putting $c = 0$ this formula reduces to formula (3) as indeed it should, being based upon it.

Putting $d = 0$ we obtain a formula for the mutual inductance of coaxial circles in the same plane.

$$
\begin{aligned}
M = 4\pi a \Bigg\{ \log \frac{8a}{c} \Bigg( & 1 + \frac{c}{2a} + \frac{c^2}{16a^2} - \frac{c^3}{32a^3} + \frac{17c^4}{8 \times 128a^4} - \frac{19c^5}{16 \times 128a^5} \\
D \quad & + \frac{89c^6}{128^2a^6} - \frac{109c^7}{2 \cdot 128^2a^7} + \frac{8{,}921c^8}{2 \cdot 128^3a^8} - \cdots \Bigg) \\
& - \Bigg( 2 + \frac{c}{2a} - \frac{3c^2}{16a^2} + \frac{c^3}{48a^3} + \frac{19c^4}{48 \cdot 128a^4} - \frac{379c^5}{480 \cdot 128a^5} \\
D' \quad & + \frac{2{,}629c^6}{30 \cdot 128^2a^6} - \frac{27{,}833c^7}{420 \cdot 128^2a^7} + \frac{5{,}204{,}309c^8}{840 \cdot 128^3a^8} - \cdots \Bigg) \Bigg\},
\end{aligned}
\tag{8}
$$

the terms in lines marked $D$ and $D'$ being new.

## Test of the Formula.

The ideal check on the new formula lies, of course, in its agreement with its derivation by independent methods such as those employed by Maxwell and by Rosa and Cohen.

But as these calculations were extremely laborious and the independent derivation by any other known method being more so, we have not undertaken to do this.

A numerical test is however of great value in such a case as this, for although it does not prove the correctness of any formula, it makes its essential correctness extremely probable.

We have made such a test for a few cases in which the true values of the mutual inductance were already known or in which they could be calculated by other absolute formulæ.

*Example I.*—Let $A$ = 25 cm., $a$ = 20 cm., $d$ = 10 cm. (see Fig. 1). The value of the mutual inductance between these two circles has been calculated by Rosa and Grover[1] by means of Maxwell's absolute formula (1) to be $M$ = 248.7875 cm.

Let $M_n$ refer to the value of the mutual inductance between these circles calculated by means of the new formula (7) carried as far as and including terms of the $n$th degree. Then the following results have been obtained.

(Rosa and Grover) $M_3$ = ———————— too large by 1 part in 1.750,
(Rosa and Grover) $M_5$ = 248.8006 cm., too large by 1 part in 19,000,
(New Formula) $M_6$ = 248.7892 cm., too large by 1 part in 125,000,
(New Formula) $M_7$ = 248.7865 cm., too small by 1 part in 250,000,
(New Formula) $M_8$ = 248.7877 cm., too large by 1 part in 1,250,000, or 4 parts in 5 million.

[1] Bulletin Bureau of Standards, 8, p. 22, example 4, 1912.

The agreement is very satisfactory and the various results show the rapidity of the convergence of the formula.

The value of $M$ computed by Havelock's formula (5) extended by means of the coefficients of formula (3) is for the same case:

$$M = 248.7873 \text{ cm.}$$

We at first thought that $M_8$ should agree exactly with the preceding value but on consideration have come to the conclusion that an *exact* agreement is not to be expected.

*Example II.*—In order to test the coefficients in formula (8), a special case of formula (7) in which $d$ has been put equal to zero, we have calculated as follows:

$$A = 15 \text{ cm.},\ a = 10 \text{ cm.},\ c = 5 \text{ cm.},\ d = 0 \text{ cm.}$$

The exact value of this mutual inductance was derived correct to about 1 part in a million, by means of the absolute formula of Maxwell, employing the tables of

$$\log \frac{M}{4\pi\sqrt{Aa}} = \left\{\left(\frac{2}{k} - k\right)F - \frac{2}{k}E\right\}$$

given in the Bulletin of the Bureau of Standards.

Its value is

$$M = 162.69605 \text{ cm.}$$

As before let $M_n$ refer to the value derived from (8) including terms of the $n$th degree, then:

$M_5 = 162.6816$ cm.; too small by 1 part in 19,000,

$M_6 = 162.6985$ cm.; too large by 1 part in 75,000,

$M_7 = 162.6943$ cm.; too small by 1 part in 100,000,

$M_8 = 162.6959$ cm.; too small by 1 part in 1,000,000.

This last result checks as nearly as the tables used will allow with the above value.

In both the above numerical cases, a very precise value would have been obtained omitting the eighth degree terms, if one half or one third of the correction due to the seventh degree terms had been used instead of its full value. There would always seem to be an improvement in doing this on account of the oscillatory character of these corrections.

## Resumé.

The important and integrable series formula of Maxwell for the mutual inductance between two coaxial circles has been extended by an independent method to include terms of the eighth degree. This is amply sufficient for the most precise calculations, in which $c/a$ and $d/a$ are not too near to unity.

The coefficients of the series including the fifth degree terms, as determined by another method by Rosa and Cohen, have been completely verified.

The correctness of the new formula has been tested by numerical examples and has been found to agree with absolute results to at least 1 part in 1,000,000, this being the limit of accuracy of the tables employed in the calculations.

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