VECTOR POTENTIAL IN RELATION TO VECTOR ANGLE

By Frederick Slate.

THE recent freer use of the various potentials has not carried with it their adoption as quantities that are physical in the fullest sense of the word; they continue to be set off somewhat as "Rechnungsgrössen"—formal aids to calculation or summarized expression, employed without always searching deep for possible physical significance. It is true that the oldest potential, the scalar potential of gravitation, electric charges and magnets, has recognized connections with energy, familiar enough in fact to guide the conventions standardizing its sign and magnitude. But the vector potential, its chief companion, is still more enveloped in mathematical haze than is desirable; its physical aspects are obscured by its association with certain operations and theorems whose mystery is not yet dissipated. As a mode of statement within modern electromagnetic theory, however, accepted finally, it would appear, after some protest against its inadequacy or superfluousness, the vector potential will be engaging increased attention. It should be properly strengthened by whatever of direct physical meaning we can gather round it, and what follows here is a small contribution to that general end, taking up a clew offered through the vector algebra, which enables us to read a new sense into the parallelism between the scalar and the vector potential, to the advantage of the latter, rated as a physical concept.

First as to the geometrical and preparatory ideas, that happen to be connected with solid angle. Assuming a pole (P) let the extremity of a radius vector $(\boldsymbol{r})$ be guided along any closed curve (s) in space, tracing in general some conical mantle, whose surface we may suppose not to cut itself, in order to avoid mere complication in details. Then the differential tangent-sectors give instantaneously the resultant elements of plane angle described by the radius vector, the aggregate of angle being a vector sum. This appears in the form

$$d\boldsymbol{\alpha}_1 = \frac{1}{r^2}[\boldsymbol{r}d\boldsymbol{s}]; \quad \int d\boldsymbol{\alpha}_1 = \int_s \frac{1}{r^2}[\boldsymbol{r}d\boldsymbol{s}], \qquad (1)$$

which includes the particular case where the guide-curve (s) lies all in one plane that contains the pole (P) also.[1] The sphere of unit radius

[1] Slate, Phys. Rev., N. S., *1*, p. 56.

(unit sphere) centered at the pole determines by its intersection with the conical mantle the boundary (s_1) of the solid angle subtended by the curve (s) at the point (P), if we accept the area of the spherical cap $(r = 1)$ as the measure of that solid angle.[1] Then by a simple continuation we write the vector expression for an areal element on the unit sphere, in the strip bordered by the curve (s_1) whose width is determined by a second coördinate $(d\boldsymbol{s}_2)$,

$$d\boldsymbol{S}_1 = [d\boldsymbol{s}_1 d\boldsymbol{s}_2] = [d\boldsymbol{\alpha}_1 d\boldsymbol{\alpha}_2]. \tag{2}$$

The third member of this equation can be read in two ways, according to the form of the thought attached to the second member. First, in terms of angle-area, the third member multiplies axial vectors; perpendiculars to the linear vectors multiplied in the second member, but numerically equal, each to each. The second factor $(d\boldsymbol{\alpha}_2)$ is the angle for the center (P) corresponding to the arc $(d\boldsymbol{s}_2)$. The product, being of necessity oriented in the radius vector, is evidently unchanged by turning both factors through any common angle about that line. But secondly, the factors in the third member are only synonyms for those in the second, if we can adopt the arc-element of the unit circle to represent angle (*i. e.*, by a linear vector in the plane of the angle).[2] Either point of view enables us to look upon solid angle as generated by an integrated double movement in plane angle of the radius vector, a scalar sum being formed finally of the elements of area whose normals are all radial. And since the cap has been taken on the unit sphere, these results again are entirely consistent with a connection of solid angle and volume, through the expression[3]

$$A = \int_S \frac{1}{r^3}(\boldsymbol{r}d\boldsymbol{S}). \tag{3}$$

Further, as a closing word on the geometrical relations, we may note that the sector of the unit sphere determined by the curve (s_1) being

[1] Following Maxwell's definition, El. and Mag., *2*, p. 39. A double or triple line of specification, whose equivalences rest finally upon certain coincidences of value for unit radius, is found current and supported by good authority through the whole group of angular quantities. There seems to be no ground in logic for adhering exclusively to one plan and ignoring special conveniences; and any necessary reassignment of dimensions can be taken account of easily. This freedom of choice among alternatives is made use of later.

[2] This presentation of angle as arc-element is not complete in vector algebra, it is clear, unless somehow supplemented by knowledge of the angle's plane; here, *e. g.*, through location of the center (P). This measure of angle underlies the specification of angular velocity as cm./sec. (explicit in Klein and Sommerfeld, Kreiseltheorie, p. 11). Such instances remind us that a vector quantity may be treated on more conventional bases than one; with "no questions asked" about pedigree.

[3] Slate, loc. cit., p. 57.

bounded by a closed surface, the projections on any plane of its conical mantle and of its spherical cap must be equal magnitudes, but subject in sign, of course, to the established rule of "outward normal." This conclusion has an evident extension to any distribution of physical vector that is homogeneously proportional to area throughout mantle and cap.[1]

Passing now to physical application of what precedes, as a matter of greater present concern, the most essential points can be brought out if we consider the simplest case: The curve (s) is a rigid conducting circuit carrying stationary current, and a positive magnet-pole is at the point (P). The inclusion of displacement current; and of conditions that introduce retarded potentials; can be shown without difficulty to demand no fundamental modifications. Reckoning then for unit of current and of pole strength (and we will add unit of permeability), the resultant interaction between the pole and a current-element in the direction ($\boldsymbol{ds}$) at an angle (ϑ) with the radius vector ($\boldsymbol{r}$) drawn from the point (P), is given correctly by

$$d\boldsymbol{F}_1 = \pm \frac{1}{r^3}[\boldsymbol{r}\boldsymbol{ds}]. \tag{4}$$

For our purpose the relations depend upon configuration, and we may regard the pole as fixed, while the circuit suffers translational displacement. The element of force-moment for origin at (P), active on the element of circuit, is

$$d\boldsymbol{M}_1 = \left[\boldsymbol{r},\ -\frac{1}{r^3}[\boldsymbol{r}\boldsymbol{ds}]\right] = d\boldsymbol{s}_1 = \frac{ds}{r}\sin\vartheta \left\{ \begin{array}{l}\text{perpendicular to } r\\ \text{in the plane } (r, ds)\end{array} \right\}. \tag{5}$$

The directed elements of the curve (s_1) on the unit sphere gain thus a new special meaning, adaptable to any general values for current and pole-strength by the introduction of a proper tensor as factor, which we shall continue here, however, to treat as unity. Moreover, our value for the element of force-moment about an axis at (P) drawn parallel to ($\boldsymbol{ds}_1$) exhibits to inspection a maximum (all other conditions remaining unaltered),

$$d\boldsymbol{M}_1(\text{max.}) \equiv d\boldsymbol{M} = \frac{1}{r}\boldsymbol{ds}\left[\text{for the direction } \vartheta = \frac{\pi}{2}\right]. \tag{6}$$

And the form of the expression in eq. (5) opens the way to consider this maximum (oriented parallel to ($\boldsymbol{ds}$)) as a resultant, in the sense that its projection upon the actual axis yields the force-moment available about

[1] Compare Maxwell, El. and Mag., 2, pp. 29–44, as a classic reference, needing only to be "popularized" by translation into simpler modern forms, and by some reading between the lines.

that axis. Therefore specially, assuming reference axes at the origin (P) and using the direction-cosines (l, m, n) of ($d\boldsymbol{s}$), we derive for moments available about (X, Y, Z):

$$dM_x = \frac{l}{r}ds;\ dM_y = \frac{m}{r}ds;\ dM_z = \frac{n}{r}ds. \tag{7}$$

Let now the circuit (s) be shifted by translation, causing angular displacement ($d\boldsymbol{\alpha}$) of the radius vector drawn to ($d\boldsymbol{s}$), and the doing of work by the force-moment ($d\boldsymbol{M}_1$) to the amount

$$dW = (d\boldsymbol{M}_1 d\boldsymbol{\alpha}) = d\boldsymbol{M}_1 d\boldsymbol{\alpha}' = (d\boldsymbol{M} d\boldsymbol{\alpha}'). \tag{8}$$

The factor ($d\boldsymbol{\alpha}'$) is the effective angular displacement; namely the component colinear with the total force-moment ($d\boldsymbol{M}_1$) of eq. (5). Hence we may read equations (7) and (8) in these terms:

The maximum force-moment ($d\boldsymbol{M}$) *gives by a definite plan of projection upon any line through the origin the work per unit angle* (*or also the work reckoned for unit angle*) *of displacement about that line as axis. And these alternatives are dimensionally equivalent, when angle is a numerical quantity.*

And further, returning to eq. (2), since the factors ($d\boldsymbol{s}_1$) and ($d\boldsymbol{\alpha}_2$) are colinear when the width ($d\boldsymbol{s}_2$) is perpendicular to ($d\boldsymbol{s}_1$) and we are dealing with angle-area, the scalar product in eq. (8) is the numerical equivalent of the vector product in eq. (2); and on comparing we can see how the work done locally by the active moment comes to be measurable by an increment ($d\boldsymbol{S}_1$) of solid angle at an element of the boundary (s_1).

But it may be desirable to think of the pole as displaceable and the circuit as fixed; and this transition offers no difficulty. For just as both aspects of an interaction in the junction-line of two active elements are comprehended in one concept of stress, so an interaction of the present type that is perpendicular to the line joining the active elements may be viewed inclusively as one couple, whose moment-vector accordingly may be drawn indifferently at either end of the radius vector. And the same work-element also is determined from each point of view, because the angular displacement retains sign and magnitude, when adjusted to the equivalent linear displacements of pole and circuit, which must be opposite as well as equal.

The attention given thus far to the differential element only of the interaction between circuit and pole does in fact reveal all the essentials of the situation. But it remains necessary to express the result of summing effects by taking account of the whole group of elements in the circuit (s). We can execute this conveniently by forming from eq. (7) the algebraic integrals

$$M_x = \int_s \frac{l}{r} ds; \; M_y = \int_s \frac{m}{r} ds; \; M_z = \int_s \frac{n}{r} ds. \tag{9}$$

These expressions for its components enable us to recognize the quantity $\boldsymbol{M} = \boldsymbol{M}_x + \boldsymbol{M}_y + \boldsymbol{M}_z$ as the vector potential at the point (P); and the same identification is deduced from eq. (6), if we write for the vector sum and its components[1]

$$\boldsymbol{M} = \int_s d\boldsymbol{M}; \; M_x = M \cos \alpha; \; M_y = M \cos \beta; \; M_z = M \cos \gamma. \tag{10}$$

It is apparent that no important difference exists between the two statements; only an interchanged order of the operations: (1) Projection; and (2) summation; plainly justified by the fundamental equivalence of resultant and its group of components as affected by the same operator. We shall note this legitimate inversion in another connection presently. The main result of our discussion might then be summarized at this stage in a definition of vector potential as a single-valued vector point-function, whose local value is determined in direction and magnitude by the total at each point of the force-moments due to other active material in the field, and described in the italics following eq. (8). We are thus put in a position to anticipate the reason why the vector potential presents itself as a convenience, where (and wherever) the field-forces, though (1) functions of the relative coördinates only, are (2) perpendicular to junction-lines and not directed along them; and how the scalar and the vector potential are supplementary in the dynamics of force-fields; primarily, we may say, as regards the types of interaction to which they apply naturally, for their equality in dimensions (those of energy) is on the surface formal and will not lead far. Nevertheless, the formulation in the above terms suggests on another side a comparison of the vector potential with the vector parameter (gradient) of the scalar potential rather than direct association of the potentials themselves. For the gradient may be specified in complete parallel, by substituting the element of (radial) force for the element of force-moment in the vector

[1] Eq. (9) is the plain equivalent of Maxwell, El. and Mag., *2*, p. 44, eq. (9); while his construction (p. 44, foot) can be based directly upon our eq. (10), which is also (for unit of current) the vector form found in the literature of electromagnetism. It is worth remarking that this type of line-integral shows the aggregate effect of some actual and limited distribution *in situ;* for instance, the total weight of a curved rope, not uniform in cross-section. To repeat such a summation would be meaningless, whereas a progressive displacement round a closed curve is capable of repetition, and may give through a line-integral work corresponding to a multiple-valued potential. Observe further that the integrals of eq. (9) may possibly vanish as moments, though the summed work of the moment-elements does not become zero. This is a counterpart of the fact that a couple (which is zero force) may contribute a finite amount of work.

potential. That is, with inversion of the more habitual process, just quoted as allowable, the gradient too can be obtained by assembling the projections of a group of elementary force-maxima. Pairing in the way indicated, the distribution of forcive throughout the field can be traced; the local value of the vector potential yields force-moment, and that of the gradient yields force; in both cases alike by the one operation of projection. It is equally true, however, as another unconstrained reading of what precedes, that the force-moment about any line is obtainable by a process akin to "axial differentiation" with respect to vector-angle applied to the vector potential; leading here to the same result as mere projection because this particular differentiation leaves dimensions unaltered when angle is a ratio.

If the vector potential gives one method of expressing work, known to be energetically equivalent to a second method where the scalar potential is used, the same amount of work for any given displacement becomes calculable in terms of force-moment exerted during a turning and also in terms of linear displacement under a straight pull. The familiar connection between a current-circuit and a properly chosen magnetic shell is readily deducible on the basis of eq. (8), the values of the potentials being harmonized by taking the conventional zero at infinity for both. But we must expect that an attempt to develop complete symmetry in the operations applied to the two potentials and their results will be disturbed by obstacles. For example, the imperfect balance of cross relations between forces and couples prevents a match to the partial determination of moment by the former with a contribution to force by the latter; couple-moments are free vectors while a force vector may be localized in its "line of action." Again the junction-line of two active elements is unique, but a coördinate perpendicular to such a line needs further definition. Consequently, when a derivative of the vector potential is connected with a force perpendicular to the coördinate used in forming the derivative, the particular perpendicular is in fact also specified. This is perhaps the closest analysis of the necessary vector quality in this potential; and it underlies as well the occurrence here of the curl operator, as causing "quadrantal advance" in direction for the derivatives that enter.

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