Dispersion Sum Rules, Interference Model, Duality, and the p-n Mass Difference*

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On the basis of new dispersion relations, an attempt has been made to derive an interference model together with the duality principle for the $\pi^- p \to \pi^0 n$ reaction. Also, the same technique has been applied to the calculation of the p-n mass difference with some success.

I. INTRODUCTION

THERE are several interesting applications of the new π-N dispersion relations obtained with a modified amplitude $(\nu^2-\mu^2)^{-\alpha}C(\nu)$, where $C(\nu)$ represents the ordinary scattering amplitude. From these relations, one can obtain several sum rules[1,2] for the Regge parameters of the P, P', and ρ trajectories, which are found to be well satisfied. Recently, the same method has also been applied to the so-called continuous-moment sum rules[3] for various other reactions, with interesting results. In this paper, we make an attempt to utilize the same technique in deriving the so-called interference model[4] with no possible double countings, and a version of the duality principle[5] for cases where only a single dominant Regge exchange is involved in the t channel (as in the $\pi^- p \to \pi^0 n$ forward reaction). One can also use the same technique for a calculation of the p-n mass difference, as we show in Sec. III.

First, let us consider the interference model, where we assume that the scattering amplitude can be well represented by a sum of t-channel Regge contributions and of s-channel resonant poles. This model has been extensively studied[6,7] for various reactions in the intermediate energy regions (say 1–3 BeV), with varying degrees of success and failure. The usefulness of such a model is, of course, due to the fact that in the intermediate energy region, one cannot really rely upon either a pure Regge pole model or a pure resonance model. However, recently this model has been criticized,[5,6] on the basis of finite-energy sum rules[8] (FESR), as possibly involving a double counting of certain Feynman diagrams. Obviously, this difficulty is intimately connected with the duality principle,[5] in which it is assumed that the t-channel Regge term is equal to a sum of the s-channel resonant poles. However, it does not necessarily follow that one may not find a modified version of the interference model which is consistent with both FESR and the duality principle. Indeed, Veneziano[9] has found a model amplitude satisfying all these criteria. Recently, Igi[10] successfully extended it to the case of π-N scattering. Although this work is very interesting, its theoretical foundation is by no means clear yet. Our purpose in this note is very modest in comparison. We want to achieve the limited goal of deriving in some sense both an interference model and a duality principle by using only well-established first principles such as analyticity and unitarity. As a matter of fact, we shall show in Sec. II that the interference model, when we modify it slightly, can be derived in some approximation, and that it appears to be compatible with the FESR and with the duality principle. Actually, one can even obtain an equation which may be interpreted as manifesting the duality principle.

We can apply the same technique to the calculation of the p-n mass difference. The traditional difficulty in this problem, as is pointed out by Harari,[11] is that we have to have one subtraction in the dispersion relation for the t_1 amplitude (see below). Therefore, one cannot calculate the p-n mass difference by the traditional method, since the subtraction constant is unknown. But by applying our new technique, we shall show that one can circumvent this difficulty. Unfortunately, we have to introduce rather arbitrary parameters in our theory, and the final result is very sensitive to their precise choice. However, the numerical calculation shows that the p-n mass difference is greatly improved over the result of the old calculations which ignore the difficulty pointed out by Harari.[11] Indeed, it is easy to obtain the correct p-n mass difference by this method, although we do not take it seriously, because of many ambiguities involved.

II. DERIVATION OF THE INTERFERENCE MODEL AND DUALITY PRINCIPLE

Let us consider[12] the crossing-odd forward pion-nucleon scattering amplitude $C^{(-)}(\nu)$ which describes

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[12] We use the same notation as that by K. Nishijima, *Fundamental Particles* (W. A. Benjamin, Inc., New York, 1963).

the charge-exchange reaction $\pi^- p \to \pi^0 n$. Since only the ρ Regge trajectory is dominant in the t channel, we have

$$C^{(-)}(\nu) \sim i\gamma\nu^{\alpha} e^{-(i/2)\pi\alpha}, \quad (\nu \to \infty) \tag{1}$$

where γ is the real residue.

Now, following the technique described elsewhere,[1] let us introduce an auxiliary function $F(\nu)$ given by

$$F(\nu) = C^{(-)}(\nu)\nu(\nu^2 - b^2)^{-\beta} e^{i\pi\beta}, \tag{2}$$

where b and β are arbitrary real constants satisfying the following conditions:

$$1 > \beta \geq \tfrac{1}{2}(1+\alpha), \quad b \geq \mu, \quad (\mu = \text{pion mass}). \tag{3}$$

We define the function $(\nu^2 - b^2)^{-\beta}$ in such a way that its cuts run on the real axis in the intervals $\infty > \nu \geq b$ and $-\infty < \nu \leq -b$, and that it assumes real positive values for $\nu = x + i\epsilon$, $(x \geq b,\ \epsilon \to +0)$ in the cut plane. Then it is easy to see that $F(z)$, as given by Eq. (2), is a crossing-even, real meromorphic function of z:

$$F(z) = F(-z), \quad F^*(z^*) = F(z). \tag{4}$$

Moreover, when we let $\nu \to \infty$, we have

$$F(\nu) \sim -\gamma\nu^{-(2\beta-1-\alpha)} \exp[\tfrac{1}{2}i\pi(2\beta - 1 - \alpha)]. \tag{5}$$

Thus, when one chooses $1 > \beta > \frac{1}{2}(1+\alpha)$, one can write an unsubtracted dispersion relation for $F(\nu)$:

$$F(\nu) = \frac{g^2}{m}\nu_B{}^3(b^2 - \nu_B{}^2)^{-\beta}\frac{1}{\nu_B{}^2 - \nu^2} + \frac{2}{\pi}\int_{\mu}^{\infty} d\nu' \frac{\nu'}{\nu'^2 - \nu^2} \operatorname{Im}F(\nu' + i\epsilon), \tag{6}$$

where $\nu_B = -\mu^2/2m$ is the position of the nucleon pole.

On the other hand, when we set $\beta = \frac{1}{2}(1+\alpha)$, we have

$$F(\nu) = -\gamma + \frac{g^2}{m}\nu_B{}^3(b^2 - \nu_B{}^2)^{-\beta}\frac{1}{\nu_B{}^2 - \nu^2} + \frac{2}{\pi}\int_{\mu}^{\infty} d\nu' \frac{\nu'}{\nu'^2 - \nu^2} \operatorname{Im}F(\nu' + i\epsilon), \tag{7}$$

where the first term, $-\gamma$, is the contribution from the infinitely large semicircle in the upper-half complex ν plane. Equation (7) is known as a dispersion relation once-subtracted at $\nu = \infty$. One may, alternatively, derive Eq. (7) from Eq. (6) by letting $\beta \to \frac{1}{2}(1+\alpha) + 0$. In this case, we cannot interchange the order of the limit and the integration. We must separate the ν' integration into two regions $\infty > \nu' \geq N$ and $N \geq \nu' \geq \mu$, where N is an arbitrary but very large number. Then, one can easily see that the first term, $-\gamma$, of Eq. (7) results from the integration region $\infty > \nu' \geq N$ when we let $\beta \to \frac{1}{2}(1+\alpha)$.

Now, Eq. (7) can be rewritten as

$$C^{(-)}(\nu) = i(1/\nu)(\nu^2 - b^2)^{(1+\alpha)/2} e^{-i\pi\alpha/2}[\gamma + K(\nu)], \tag{8a}$$

where

$$K(\nu) = -\frac{g^2}{m}\nu_B{}^3(b^2 - \nu_B{}^2)^{-(1+\alpha)/2}\frac{1}{\nu_B{}^2 - \nu^2} - \frac{2}{\pi}\int_{\mu}^{b} d\nu' \frac{\nu'^2}{\nu'^2 - \nu^2}(b^2 - \nu'^2)^{-(1+\alpha)/2} \operatorname{Im}C^{(-)}(\nu') - \frac{2}{\pi}\int_{b}^{\infty} d\nu' \frac{\nu'^2}{\nu'^2 - \nu^2}(\nu'^2 - b^2)^{-(1+\alpha)/2} \times \operatorname{Re}[C^{(-)}(\nu')e^{i\pi\alpha/2}]. \tag{8b}$$

Then, it is clear that the first term of Eq. (8a) gives the correct asymptotic behavior [as in Eq. (1)] for $\nu \to \infty$, so that we must have $K(\nu) \to 0$ for $\nu \to \infty$. Therefore, Eq. (8) represents a separation of the ρ-Regge term from the background.

Before going into further details, we notice that we have introduced an artificial pole at $\nu = 0$ in Eq. (8a). Since such a pole should not appear in the amplitude $C^{(-)}(\nu)$, we must have a constraint condition:

$$\gamma = -K(0) = \frac{g^2}{m}\nu_B(b^2 - \nu_B{}^2)^{-(1+\alpha)/2} + \frac{2}{\pi}\int_{b}^{\infty} d\nu' (\nu'^2 - b^2)^{-(1+\alpha)/2} \operatorname{Re}[C^{(-)}(\nu')e^{i\pi\alpha/2}] + \frac{2}{\pi}\int_{\mu}^{b} d\nu' (b^2 - \nu'^2)^{-(1+\alpha)/2} \operatorname{Im}C^{(-)}(\nu'). \tag{9}$$

This relation must be valid for arbitrary values of b satisfying the condition $b \geq \mu$. When one chooses $b = \mu$, this reduces to the sum rule of Olsson,[2] who found that it gives a good value for the ρ Regge residue γ.

Similarly, if we start from Eq. (6) with $\nu = 0$ and $b = \mu$, we have

$$\frac{g}{m}\nu_B(\mu^2 - \nu_B{}^2)^{-\beta} + \frac{2^2}{\pi}\int_{\mu}^{\infty} d\nu' (\nu'^2 - \mu^2)^{-\beta} \times \operatorname{Im}[C^{(-)}(\nu')e^{i\pi\beta}] = 0$$

for all $1 > \beta > \frac{1}{2}(1+\alpha)$, since $F(0) = 0$ by Eq. (2). This is nothing but the superconvergent dispersion relation investigated and verified in Ref. (1). Hence, Eqs. (6) and (7) contain all previous results as special cases.

So far, b is arbitrary, apart from the restriction $b \geq \mu$. If we wish to maximize the contribution of the Regge-pole term, then it is natural to choose $b = \mu$, as we see from Eq. (8a). Actually, the precise value of b should be immaterial. Indeed, for the practical calculations we make, the final results are insensitive to the value of b. Hence, in the following we shall restrict ourselves to the case $b = \mu$ for simplicity.

Coming back now to the original problem, the asymptotic behavior of $K(\nu)$ depends upon further assumptions. For instance, if the Regge cut due to the simultaneous exchange of the ρ and Pomeranchuk trajectories is important, we expect to have

$$K(\nu) \sim [a/\ln\nu + a'/(\ln\nu)^2], \quad \nu \to \infty,$$

where a is a real constant, so that the integral of Eq. (8a) exists at $\nu'=\infty$. On the other hand, if the so-called ρ' trajectory exists, we have instead

$$K(\nu)\sim\text{const}\ \nu^{-(\alpha-\alpha')}e^{(-i\pi/2)(\alpha-\alpha')}\,, \tag{10}$$

where α' is the Regge trajectory intercept at $t=0$ of the assumed ρ' meson. For this case, one can explicitly separate the asymptotic ρ' Regge contribution from $K(\nu)$ by repeating exactly the same argument as in the previous case of the separation of the ρ-Regge contribution from $C^{(-)}(\nu)$. However, this procedure is rather complicated and cumbersome, and we do not attempt it. Instead, we are interested in the possibility of expressing $K(\nu)$ as a sum of direct-channel resonance terms. To that end, let us suppose for a moment that $C^{(-)}(\nu)$ has only the elastic unitarity cut (i.e., we neglect all inelastic cuts). One can then analytically continue $K(\nu)$ into the second Riemann sheet by distorting the integration contour in Eq. (8b). Furthermore, if the asymptotic form of $C^{(-)}(\nu)$ for $\nu\to\infty$ is more rapidly decreasing than (or at most is the same as) in Eq. (1), even in the lower half of the second Riemann sheet, then one can rotate the unitarity cut by 180° in the second Riemann sheet by a well-known method.[13] Since $C^{(-)}(\nu)$ should have resonant poles at $\nu=\nu_n$ ($\mathrm{Im}\nu_n<0$) in the lower half of the second Riemann sheet, one finally obtains

$$C^{(-)}(\nu)=i\gamma\frac{1}{\nu}(\nu^2-\mu^2)^{(1+\alpha)/2}\exp(-\tfrac{1}{2}i\pi\alpha)+\frac{1}{\nu}(\nu^2-\mu^2)^{(1+\alpha)/2}\sum_n\frac{\beta_n}{\nu_n{}^2-\nu^2}\frac{\nu_n}{(\nu_n{}^2-\mu^2)^{(1+\alpha)/2}}+R(\nu)\,, \tag{11}$$

where $R(\nu)$ is the residual integral over the interval $-\infty<\nu'\leq\mu$ in the second Riemann sheet. For simplicity, we have included the Born term in the summation over n in Eq. (11). Note that at the position of the pole $\nu=\nu_n$, it has exactly the same Breit-Wigner form as an ordinary amplitude. If the residual integral $R(\nu)$ can be neglected, then Eq. (11) is nothing but the interference model,[4] apart from the multiplicative factor $(\nu^2-\mu^2)^{(1+\alpha)/2}$.

In the argument given above, we have neglected contributions from the inelastic cuts of $C^{(-)}(\nu)$ in the complex ν plane. In principle, one could repeat the same argument if one could analytically continue into the second Riemann sheet of these inelastic cuts. Then formally we would still have the formula Eq. (11). But in that case, the background integral $R(\nu)$ would consist of a sum of principal-value integrals involving a factor $(\nu'^2-\nu^2)^{-1}$ in the interval $-\infty<\nu'<\bar{\nu}$, where $\bar{\nu}$ is the threshold of the inelastic cut under consideration. The trouble now is that $\bar{\nu}$ can be as large as one likes because of many pion-production thresholds. Hence, there will be no reason why $R(\nu)$ should be negligible, unless the nonresonant inelastic cross sections are relatively small in the energy region under consideration. Therefore, one of our criteria for the validity of the interference model is that the nonresonant inelastic cross sections must be small.

One possible weak point of our derivation of Eq. (11) is the assumption that $C^{(-)}(\nu)$ must have an asymptotic behavior which is the same as or less rapid than that in Eq. (1) in the second Riemann sheet. Unfortunately, there is no *a priori* reason why this should be correct. It could be worse. However, we still expect Eq. (11) to be approximately valid for the following reason. If we approximate $C^{(-)}(\nu')$ in the integrand of Eq. (8b) by a sum of Breit-Wigner terms, then Eq. (11) emerges again approximately, irrespective of the previous argument of going into the second Riemann sheet. Such an approximation would be reasonable if the low-energy region gives the dominant contribution to the integral in Eq. (8b). Thus, we would expect Eq. (11) to be valid in either case. However, one thing that is not so clear at this point is the possible existence of resonant-like peaks which may not correspond to poles in the second Riemann sheet. In the second argument given above, such contributions must appear in the summation of Eq. (11), in addition to the legitimate poles. In view of the previous arguments stated after Eq. (11), it is quite conceivable that such peaks could result from the background integral $R(\nu)$ in the second Riemann sheet corresponding to inelastic cuts. If this should be the case, Eq. (11) must be accepted as a semiphenomenological expression rather than as a straightforward theoretical derivation. Indeed a similar idea has already been expressed by some authors.[14]

Now, before going into the question of FESR, we consider the duality principle. For this purpose, we first note that the ordinary unsubtracted dispersion relation for $C^{(-)}(\nu)$ must be valid because of Eq. (1) with $\alpha<1$:

$$C^{(-)}(\nu)=\frac{g^2}{m}\nu_B\frac{\nu}{\nu_B{}^2-\nu^2}+\frac{2\nu}{\pi}\int_\mu^\infty d\nu'\frac{1}{\nu'^2-\nu^2}\,\mathrm{Im}C^{(-)}(\nu')\,. \tag{12}$$

Comparing this with Eq. (8), one obtains the following identity:

$$i\gamma\frac{1}{\nu}(\nu^2-\mu^2)^{(1+\alpha)/2}\exp(-\tfrac{1}{2}i\pi\alpha)=B(\nu)+\frac{2\nu}{\pi}\int_\mu^\infty d\nu'\frac{\nu'^2}{\nu'^2-\nu^2}\left\{\frac{\mathrm{Im}C^{(-)}(\nu')}{\nu'^2}+i\left(\frac{\nu^2-\mu^2}{\nu'^2-\mu^2}\right)^{(1+\alpha)/2}\frac{\exp[(-\frac{1}{2})i\pi\alpha]}{\nu^2}\times\mathrm{Re}[C^{(-)}(\nu')e^{i\pi\alpha/2}]\right\}\,, \tag{13}$$

[13] E.g., R. Jacob and R. G. Sachs, Phys. Rev. **121**, 350 (1961).

[14] V. A. Alessandrini, D. Amati, and E. J. Squires, Phys. Letters **27B**, 463 (1968).

where we have set $b=\mu$; the Born term $B(\nu)$ is given by

$$B(\nu)=\frac{g^2}{m}\frac{\nu}{\nu_B{}^2-\nu^2}\nu_B \times\left[1+\left(\frac{\nu_B}{\nu}\right)^2\left(\frac{\nu^2-\mu^2}{\mu^2-\nu_B{}^2}\right)^{(1+\alpha)/2} i\exp(-\tfrac{1}{2}i\pi\alpha)\right]. \quad (14)$$

Notice that the left-hand side of Eq. (13) is purely the ρ-Regge contribution, while the right-hand side of Eq. (13) contains quantities involving the s channel, independent of the Regge residue γ. In this sense, we may regard Eq. (14) as a manifestation of the duality principle.[5] If we analytically continue the right-hand-side integral of Eq. (13) into the second Riemann sheet as before, then we can rewrite it as follows:

$$i\gamma\frac{(\nu^2-\mu^2)^{(1+\alpha)/2}}{\nu}\exp(-\tfrac{1}{2}i\pi\alpha) =-\sum_n\frac{\beta_n}{\nu_n{}^2-\nu^2}\left[\frac{\nu_n}{\nu}\left(\frac{\nu^2-\mu^2}{\nu_n{}^2-\mu^2}\right)^{(1+\alpha)/2}-\frac{\nu}{\nu_n}\right]+R'(\nu), \quad (15)$$

where we have incorporated the Born term in the summation over n, and where $R'(\nu)$ is the residual background integral as before. In this form, the t-channel ρ-Regge term is now expressed as a sum of s-channel resonant poles, if one can neglect $R'(\nu)$. Note that the right-hand side of Eq. (15) is now a smooth function of ν without resonant peaks at $\nu=\mathrm{Re}(\nu_n)$. One can numerically compare both sides of Eq. (15)—neglecting $R'(\nu)$. If we take into account only contributions from a few low-lying resonances listed in the Rosenfeld Table,[15] then we find that the left-hand side of Eq. (15) gives values much larger than the right-hand side, although the signs of both sides are the same. This trend is worse for the imaginary part by a factor of 10, especially in the high-energy region, ($\nu\geq 3$ BeV). For the real part, the discrepancy is by a factor of 2 in the energy region 3 BeV$\geq\nu\geq$1 BeV. This fact seems to suggest the following alternatives: Either $R'(\nu)$ is not negligible at all, or we have to take into account slowly convergent infinite sequences of resonant poles, which may correspond to Regge recurrences of the N and Δ trajectories and possibly of their daughters. In the first case, our formula Eq. (15) is completely useless since we have no reliable way of calculating $R'(\nu)$. The second case obviously corresponds to the duality principle in the ordinary sense. In the latter case unfortunately, we cannot use the narrow-width approximation since the imaginary part of the right-hand side will vanish in that approximation. As a matter of fact, this is the practical reason why the imaginary part of the right-hand side of Eq. (15) is so small for a few low-lying levels. But then, because of the many successes of the FESR in conjunction with the narrow-width approximation, this fact appears to make our interpretation a bit implausible. However, this is the only alternative reconciling Eq. (15) with the idea of the duality principle. Suppose that on the contrary $R'(\nu)$ is important and not negligible. Then we have to conclude that the t-channel ρ-Regge pole cannot be built up from the s-channel resonant poles alone. Hence, we believe that the narrow-width approximation somehow does not work for Eq. (15). Of course, one may argue that Eq. (15) is not valid, probably because of the impossibility of analytically continuing $C^{(-)}(\nu)$ into the second Riemann sheet. Our dilemma will then disappear, but we have nothing to gain from this possibility. At any rate, if we accept our philosophy as stated above, then our modified interference model [Eq. (11)] is consistent with the FESR. Note that the FESR is written as

$$\int_0^N d\nu\,\mathrm{Im}C^{(-)}(\nu)=\gamma\frac{1}{\alpha+1}\cos\tfrac{1}{2}\pi\alpha\,N^{\alpha+1}. \quad (16)$$

Because of Eq. (15), our formula Eq. (11) is now rewritten as

$$C^{(-)}(\nu)=\sum_n\frac{\beta_n}{\nu_n}\frac{\nu}{\nu_n{}^2-\nu^2}+R''(\nu), \quad (17)$$

which is nothing but a modified way of writing Eq. (12). Then Eqs. (16) and (17) constitute the content of the ordinary FESR. In this connection, it may be worthwhile to derive the modified interference model of Dolen, Horn, and Schmid[5] in an analogous way. For this purpose, we start from a new dispersion relation

$$C^{(-)}(\nu)=i\gamma\nu(\nu^2-\mu^2)^{(\alpha-1)/2} \times\exp(-\tfrac{1}{2}i\pi\alpha)+\frac{2\nu}{\pi}\int_\mu^\infty d\nu'\frac{1}{\nu'^2-\nu^2} \times\mathrm{Im}[C^{(-)}(\nu')-i\gamma\nu'(\nu'^2-\mu^2)^{(\alpha-1)/2}\exp(-\tfrac{1}{2}i\pi\alpha)]. \quad (18)$$

Experimentally, the integrand of Eq. (18) oscillates around each resonant pole $\nu'=\mathrm{Re}\nu_n$. Hence, one can approximate it by $\mathrm{Im}f_{\mathrm{Res}}(\nu')-\mathrm{Im}\langle f_{\mathrm{Res}}(\nu')\rangle$, where $\langle f_{\mathrm{Res}}(\nu')\rangle$ implies some kind of averaging over resonant peaks. Integrating over ν', one finds that Eq. (18) leads to

$$C^{(-)}(\nu)=f_{\mathrm{Regge}}(\nu)+f_{\mathrm{Resonance}}(\nu) -\langle f_{\mathrm{Resonance}}(\nu)\rangle+R''(\nu), \quad (19)$$

which, if we neglect $R''(\nu)$, is nothing but the modified interference model due to Dolen, Horn, and Schmid.[5] We remark that in this case, the analytic continuation method into the second Riemann sheet is probably not so good, because now the residual integral still contains the Regge asymptotic term. Otherwise, we would have some conflicts with Eq. (11). This may imply that

[15] N. Barash-Schmidt, A. Barbaro-Galtieri, L. R. Price, A. H. Rosenfeld, P. Söding, G. Wohl, and M. Roos, CERN Report, 1968 (unpublished).

$C^{(-)}(\nu)$ decreases much more rapidly in the second Riemann sheet than implied by Eq. (1).

In conclusion, we have been able to give some rationale for the validity of interference model and the duality principle. But our discussion should not be taken as a proof of this validity. As we have seen, we have to make many additional assumptions, some of which are by no means obvious or may even be a bit implausible. Even if we accept our derivation, it is valid only for the case where a single Regge exchange is dominant. For the multi-Regge exchange case, one can, in principle, repeat the same argument. But as we emphasized already, the result is too complicated to allow any simple interpretation. Thus, at least for $\pi^- p \to \pi^- p$ in the backward direction, and for $\pi^- p \to \pi^0 n$ in the forward direction, we may apply our formalism, since these reactions correspond to a single dominant Regge exchange in the t or u channel. Indeed, according Chiu and Stirling,[6] the interference model may account reasonably well for these reactions. Also, the presence of an extra factor $(\nu^2-\nu_n{}^2)^{(1+\alpha)/2}$ in our Eq. (11) can help to give a larger polarization in $\pi^- p \to \pi^0 n$ reaction. We have made a calculation at this point and obtain a polarization as large as 5–7% in the range $0.05<|t|<0.35$ to 5 BeV, and on average of 4% at 11 BeV in the same range of t. With respect to the total and differential cross sections, our result is more or less similar to more exhaustive calculations given by Chiu and Stirling,[6] and we have nothing to add. A generalization of our technique is given in the Appendix.

III. p-n MASS DIFFERENCES

We start with the Cottingham formula[16]

$$\Delta m=-\frac{1}{2\pi}\int_0^\infty \frac{d(q^2)}{q^2}\int_0^q d\nu(q^2-\nu^2)^{1/2} \times[3q^2t_1(q^2,i\nu)-(q^2+2\nu^2)t_2(q^2,i\nu)]. \quad (20)$$

If one can use unsubtracted dispersion relation for t_1 and t_2, then one has for $i=1, 2$

$$t_i(q^2,\nu)=\frac{4mq^2}{q^4-4m^2\nu^2}f_i(q^2)+\frac{2}{\pi}\int_C^\infty \frac{\mathrm{Im}t_i(q^2,\nu')}{\nu'^2-\nu^2}\nu' d\nu', \quad (21)$$

where

$$\begin{aligned} C&=(1/2m)(2m\mu+\mu^2+q^2),\\ f_1(q^2)&=(\alpha_0/\pi)(q^2+4m^2)^{-1}[G_M{}^2(q^2)-G_E{}^2(q^2)]_{I=1}, \quad (\alpha_0=e^2/\hbar c), \quad (22)\\ f_2(q^2)&=(\alpha_0/\pi)[q^2(q^2+4m^2)]^{-1} \times[q^2G_M{}^2(q^2)+4m^2G_E{}^2(q^2)]_{I=1}. \end{aligned}$$

However, Harari[11] has pointed out that if we assume the validity of the Regge asymptotic behavior, we would have the following asymptotic behavior for t_i, when we let $\nu\to\infty$

$$\begin{aligned} t_1(q^2,\nu)&\sim\gamma_1(q^2)\nu^\alpha \exp(-\tfrac{1}{2}i\pi\alpha), \quad \nu\to\infty\\ \nu^2 t_2(q^2,\nu)&\sim\gamma_2(q^2)\nu^\alpha \exp(-\tfrac{1}{2}i\pi\alpha), \quad \nu\to\infty \end{aligned} \quad (23)$$

where $\alpha=\alpha_{A_2}(0)$ is the $t=0$ intercept of the A_2-Regge trajectory and it is, independent of q^2. Therefore, we cannot use an unsubtracted dispersion relation for t_1, although we could do so for t_2. However, if we use a once-subtracted dispersion relation for t_1, then this introduces an unknown constant, and we cannot calculate the p-n mass difference. But one can overcome this difficulty by using a method analogous to that used in Sec. II.

Let us consider an amplitude given by

$$G(\nu,q^2)=t_1(q^2,\nu)(\nu^2-b^2)^{-\beta}\exp(i\pi\beta) \quad (24)$$

analogous to Eq. (2), where we choose

$$1>\beta\geq\tfrac{1}{2}\alpha, \quad b>q^2/2m, \quad (25)$$

but now, we shall not impose the condition $b\geq c$. We repeat the same argument for $G(\nu,q^2)$, obtaining the following equation:

$$\begin{aligned} t_1(\nu,q^2)=R(q^2)&+\frac{4mq^2}{q^4-4m^2\nu^2} \times f_1(q^2)\left(b^2-\frac{q^4}{4m^2}\right)^{-\beta}(\nu^2-b^2)^\beta\exp(-i\pi\beta)\\ &+\frac{2}{\pi}(\nu^2-b^2)^\beta\exp[-i\pi\beta]\int_d^\infty d\nu'\frac{\nu'}{\nu'^2-\nu^2} \times\mathrm{Im}[t_1(\nu',q^2)(\nu'^2-b^2)^{-\beta}\exp(i\pi\beta)], \quad (26) \end{aligned}$$

where d and $R(q^2)$ are given by

$$\begin{aligned} d&=\min(b,c),\\ R(q^2)&=0 \quad \text{for } 1>\beta>\tfrac{1}{2}\alpha,\\ &=\gamma_1(q^2)(\nu^2-b^2)^{\alpha/2}\exp(-\tfrac{1}{2}i\pi\alpha) \quad \text{for } \beta=\tfrac{1}{2}\alpha. \quad (27) \end{aligned}$$

Note that $R(q^2)$ is a discontinuous function of β with the discontinuity at $\beta=\frac{1}{2}\alpha$. This again implies [see the argument following Eq. (7) of Sec. II] the impossibility of interchange of the order of the limit $\beta\to\frac{1}{2}\alpha+0$ and the integral. (This example may be taken as a warning against the indiscriminate use of the so-called Bjorken limiting procedure where interchanges of the limit $q_0\to\infty$ and integrals are freely assumed.)

If we choose $\beta=\frac{1}{2}\alpha$, then the first term $R(q^2)$ in Eq. (26) corresponds to the contribution of the A_2 Regge trajectory, and it will lead to the Reggeized tadpole model, which has been discussed by several authors.[17] To discover the explicit form of $\gamma_1(q^2)$, one can appeal to the finite-energy sum rule, as Srivastava[17] has done. In this way, he obtained a large negative contribution for Δm ($\Delta m\approx-2.5$ BeV). However, the same calculation has been recently repeated by several other

[16] W. N. Cottingham, Ann. Phys. (N. Y.) 25, 424 (1963); M. Cini, E. Ferrari, and R. Gatto, Phys. Rev. Letters 2, 7 (1959).

TABLE I. Calculation of Δm with $b=c$.

β	y	z (BeV^2)	Δm (MeV)
$\frac{1}{3}$	0.04	40	-1.2
$\frac{1}{2}$	0.04	30	-1.3
$\frac{2}{3}$	0.04	20	-1.1
$\frac{2}{3}$	0		$+0.32$
$\frac{3}{4}$	0		$+0.28$

TABLE II. Calculation of Δm, with $y=0$.

β	λ	Δm (MeV)
$\frac{2}{3}$	1	-0.62
$\frac{2}{3}$	0.5	-1.32
$\frac{3}{4}$	0.5	-2.53
$\frac{2}{3}$	5.0	$+0.14$
$\frac{3}{4}$	5.0	-0.01

authors[18] and it has been found in contrast that its contribution to Δm is rather small—of the order -0.3 MeV at most. A similar estimate has also been obtained by Hara[19] who used the field-current identity and $SU(6)$ theory to evaluate $\gamma_1(q^2)$. Hence, it appears that the Reggeized tadpole term gives too small a contribution to account for the p-n mass difference. However, it must be borne in mind that the value of Δm is extremely sensitive to the behavior of $\gamma_1(q^2)$, for $q^2\to\infty$, as is shown shortly, and that there is really no reliable way to determine its precise asymptotic behavior.

In this paper, we avoid computing the contribution from the Reggeized tadpole term by choosing $\beta>\frac{1}{2}\alpha\approx\frac{1}{6}$. Also, we calculate the contribution to Δm only from the Born term [the second term in Eq. (26)], thus neglecting the continuum contributions. For the precise form of G_E and G_M, we use the following parametrization:

$$\begin{aligned}(1/\mu_p)G_M{}^{(p)}(q^2)&=G_E{}^{(p)}(q^2)=[(1+q^2/0.71)^2]^{-1},\\(1/\mu_n)G_M{}^{(n)}(q^2)&=(4m^2/q^2\mu_n)G_E{}^{(n)}(q^2)\\&=[(1-y)/(1+q^2/0.71)^2]+y/(1+q^2/z)^2.\end{aligned}\tag{28}$$

The standard form[20] corresponds to the choice $y=0$. However, Δm is extremely sensitive to the asymptotic form of $G(q^2)$ when $q^2\to\infty$. Also, it is sensitive to the value of b and β. In general, if we choose as small a b and as large a β as possible, the calculated Δm improves. As an example, let us choose $b=c$. The result is summarized in Table I. For comparison, we note that the old calculation, corresponding to $\beta=0$ and $y=0$, will give $\Delta m=+0.6$ MeV. We remark that the values of y and z given in Table I are consistent[21] with the observed $G(q^2)$.

To see the dependence of Δm upon b, we choose the following parametrization:

$$b^2=q^4/4m^2+[(2m\mu+\mu^2)^2/4m^2]\lambda\quad(\lambda>0).\tag{29}$$

Setting $y=0$ in Eq. (28), one obtains the values for Δm summarized in Table II.

As we see from these tables, the calculated Δm is much improved, compared with the old calculation. Also, Δm is extremely sensitive to the precise asymptotic form of $G(q^2)$ and to values of b and β. Indeed, in several instances, one could easily reproduce the experimental value for Δm. However, because of the sensitivity of Δm to various parameters, such agreement cannot be taken seriously. Besides, the neglect of the continuum integral may not be justifiable.[22] We gave our calculation to indicate a possibility of explaining the observed p-n mass difference. This remark also applies to the calculations of Reggeized tadpole contributions. Using a sum rule for $\gamma_1(q^2)$ which can be obtained by equating the two equations of Eq. (26), corresponding to the choices $\beta>\frac{1}{2}\alpha$ and $\beta=\frac{1}{2}\alpha$, respectively, one can estimate its contribution to Δm. Again neglecting the continuum integrals, one finds that the result is extremely sensitive[23] to the asymptotic form of $G(q^2)$ for $q^2\to\infty$ and to the values of b and β. Therefore, the small values found by various authors[18,19] must be taken with some caution. In our opinion, it is still possible that the Reggeized tadpole term could be the dominant contribution,[24] as was originally postulated.

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APPENDIX

Here, we briefly consider a generalization of the dispersion relations used in Secs. II and III.

We consider arbitrary functions $\varphi^{(\pm)}(\nu)$ satisfying the following conditions:

[17] S. Okubo, Phys. Rev. Letters **18**, 256 (1967); Y. C. Liu and S. Okubo, Nuovo Cimento **52A**, 1186 (1967); G. Segrè, Phys. Rev. Letters **20**, 29 (1968); Y. Srivastava, Phys. Rev. Letters **20**, 232 (1968).

[18] K. Miura and T. Watanabe, Tokyo University of Education Report, 1968 (unpublished); R. Rockmore (private communication).

[19] Y. Hara, Tokyo University of Education Report, 1968 (unpublished).

[20] See, for example, S. Gasiorowicz, *Elementary Particle Physics* (Wiley-Interscience, Inc., New York, 1966).

[21] J. R. Dunning, Jr., K. W. Chen, A. A. Cone, G. Hartwig, N. F. Ramsey, J. K. Walker, and R. Wilson, Phys. Rev. **141**, 1286 (1966); W. K. H. Panofsky, in *Proceedings of the Heidelberg Conference* (North-Holland Publishing Co., Amsterdam, 1968).

[22] However, some calculations indicate that the contribution from the second N^* resonance may be small. See T. Muta, University of Sussex Report, 1968 (unpublished); R. Rockmore, Rutgers University Report, 1968 (unpublished).

[23] M. Suzuki, Phys. Rev. **173**, 1473 (1968), has proposed an analytic-continuation method for the Regge parameter α when the q^2 integral at infinity formally diverges. In this case also, the result depends quite sensitively upon the asymptotic form of $\gamma_1(q^2)$ when $q^2\to\infty$, in conformity with our claim.

[24] J. S. Ball and F. Zachariasen, Phys. Rev. **177**, 2264 (1969).

(i) crossing property: $\varphi^{(\pm)}(\nu)=\pm\varphi^{(\pm)}(-\nu)$.

(ii) reality condition: $[\varphi^{(\pm)}(\nu)]^*=\varphi^{(\pm)}(\nu^*)$.

(iii) analyticity conditions: $\varphi^{(\pm)}(\nu)$ is meromorphic with cuts on the real axis for $\nu\geq a$ and $\nu\leq -a$, where a is an arbitrary real constant satisfying $a\geq\mu$. $\varphi^{(\pm)}(\nu)$ may have poles in the cut plane at $\nu=\nu_n$, with residue β_n. Because of (ii) and (iii) we note that $\mathrm{Im}\varphi^{(\pm)}(\nu)=0$ on the real interval $-a<\nu<a$, except for possible δ-function contributions from its poles.

(iv) Asymptotic behavior:

$$\begin{aligned}\varphi^{(-)}(\nu)&\sim -i\lambda_-\nu^{-\tau}e^{i\pi\tau/2},\\ \varphi^{(+)}(\nu)&\sim -\lambda_+\nu^{-\tau}e^{i\pi\tau/2},\end{aligned}\tag{A1}$$

for $\nu\to\infty$ in the cut plane; where $\lambda_\pm$ and τ must be real because of (i) and (ii).

Now consider a function

$$G^{(\pm)}(\nu)=C^{(\pm)}(\nu)\varphi^{(\pm)}(\nu).\tag{A2}$$

Then, $G^{(\pm)}(\nu)$ is a crossing-even function of ν, while all the other requirements (ii)–(iv) are satisfied. Setting

$$\begin{aligned}C^{(+)}(\nu)&\sim -\gamma_+\nu^\alpha\exp(-\tfrac{1}{2}i\pi\alpha),\\ C^{(-)}(\nu)&\sim +i\gamma_-\nu^\alpha\exp(-\tfrac{1}{2}i\pi\alpha),\quad(\nu\to\infty)\end{aligned}\tag{A3}$$

one finds

$$G^{(\pm)}(\nu)\sim +\gamma_\pm\lambda_\pm\nu^{-(\tau-\alpha)}\exp[i\pi(\tau-\alpha)/2]$$

for $\nu\to\infty$. Hence, one can find the following dispersion relations:

$$\begin{aligned}G^{(\pm)}(\nu)=R+\frac{g^2}{2m}\varphi^{(\pm)}(\nu_B)\frac{2\nu_B}{\nu_B{}^2-\nu^2}+\sum_n\frac{\beta\nu}{\nu_n-\nu}C^{(\pm)}(\nu_n)\\ +\frac{2}{\pi}\int_\mu^\infty d\nu'\frac{\nu'}{\nu'^2-\nu^2}\,\mathrm{Im}G^{(\pm)}(\nu'+i\epsilon),\end{aligned}\tag{A4}$$

where R is given by

$$\begin{aligned}R&=0 &&\text{if}\quad \tau>\alpha,\\ &=\lambda_\pm\gamma_\pm &&\text{if}\quad \tau=\alpha.\end{aligned}\tag{A5}$$

Then the result of Sec. II corresponds to the choice

$$\varphi^{(-)}(\nu)=(\nu^2-b^2)^{-\beta}e^{i\pi\beta}.$$

One can repeat essentially the same argument of Secs. II and III on the basis of Eq. (A4).

One can also obtain the following superconvergent dispersion relations in a similar manner:

$$\begin{aligned}\int_a^\infty d\nu\,\mathrm{Im}[\varphi^{(\mp)}(\nu+i\epsilon)C^{(\pm)}(\nu+i\epsilon)]\\ +\int_\mu^a d\nu\,\varphi^{(\mp)}(\nu+i\epsilon)\,\mathrm{Im}C^{(\pm)}(\nu+i\epsilon)\\ =\pi\left(\frac{\mu}{2m}g\right)^2\varphi^{(\mp)}(\nu_B)+\frac{\pi}{2}\sum_n\beta_n\,\mathrm{Re}C^{(\pm)}(\nu_n)+\tfrac{1}{2}\pi R,\end{aligned}\tag{A6}$$

depending upon $\tau>\alpha$ and $\tau=\alpha$ exactly as in Eqs. (A4) and (A5). Note that we are now using the crossing-odd combination $\varphi^{(\mp)}(\nu)C^{(\pm)}(\nu)$ instead of the crossing-even combination $\varphi^{(\pm)}(\nu)C^{(\pm)}(\nu)$. These sum rules are generalizations of those of Sec. II and of Ref. 1.

As an application of this sum rule, we may set

$$\varphi^{(-)}(\nu)=\nu(a^2-\nu^2)^{1/2}/(\nu^2+b^2)^2,\tag{A7}$$

where b is an arbitrary real constant. Then $\varphi^{(-)}(\nu)$ has poles at $\nu=\pm ib$. In Eq. (A6), $C^{(+)}(\pm ib)$ can be calculated by the use of the ordinary once-subtracted dispersion relation. Then by means of Eq. (A6) the Pomeranchuk residue γ_+ can be calculated using only experimentally measurable quantities. An advantage of using Eq. (A7) is that the integrand of Eq. (A6) is now a continuous, bounded function of ν. Therefore, our evaluation of γ is less sensitive to the precise knowledge of $C^{(\pm)}(\nu)$ at $\nu=\mu$, as has been encountered in the Ref. 1. Actually, we estimate $C^{(+)}(\nu)$ up to 8 BeV by using the data of Hohler *et al.*, and up to 29 BeV by using that of Foley *et al.*[25] For $\nu>29$ BeV, we use the Regge asymptotic formula Eq. (3). In this way, we find that the value of $\gamma_+=\gamma_P$, with $\alpha=1$, is given by

$$\gamma_P\equiv\sigma_\pm(\infty)=23.5\text{ mb}$$

consistently for all values of b from 0 to 10 BeV; here we have used $a=29$ BeV. This value is very near to that found in Ref. 1.

Similarly, by choosing

$$\varphi^{(+)}(\nu)=(\nu^2+b^2)^{-1}(a^2-\nu^2)^{(3-\alpha_\rho)/2}$$

one can estimate $\gamma_\rho\sim 7.5$ mb BeV, where we assumed $\alpha_\rho=0.58$. This result agrees with that obtained by Olsson.[2] The details of the calculation, together with other applications to KN, $\pi\pi$, and $K\pi$ scatterings, will be given elsewhere.

[25] G. Höhler, G. Ebel, and J. Giesecke, Z. Physik. **180**, 430 (1964); K. J. Foley *et al.*, Phys. Rev. Letters **19**, 193 (1967); **19**, 330 (1967).