Holomorphy and Phase Transitions: A Scaling Law for Magnetic Systems

M. H. COOPERSMITH

Physics Department, Case Western Reserve University, Cleveland, Ohio

(Received 25 September 1967)

A new thermodynamic inequality involving two hitherto unmeasured critical exponents for magnetic systems is derived. A class of thermodynamic potentials is also presented whose critical exponents satisfy this inequality and Rushbrooke's inequality with the equality sign (scaling law), but satisfy Griffith's inequality with the inequality sign as well.

IN a recent note,[1] we exhibited a class of thermodynamic functions representing potentials which are holomorphic in the applied magnetic field H and reduced temperature $t=(T-T_c)/T_c$, in a neighborhood of the critical point of a ferromagnet. The functions presented were algebraic functions of H and t which were homogeneous in H^{2x} and t^y, where x and y are two arbitrary positive integers. These thermodynamic potentials led to the equality sign in the various known thermodynamic inequalities relating critical exponents. In particular, the following two inequalities, due to Rushbrooke[2] and Griffiths,[3] respectively, are satisfied with the equality sign by these functions.

$$\alpha'+2\beta+\gamma'\geq 2, \tag{1}$$

$$\alpha'+\beta(\delta+1)\geq 2. \tag{2}$$

Upon further examination of the homogeneous algebraic functions, we have discovered another scaling-law equality which has much the same status as Eq. (1) and is almost as amenable to experimental verification. This equality arises in the following way. We define the homogeneous algebraic function, $F(H,t)$ by the equation

$$P=F^n+a_1(H,t)F^{n-1}+\cdots+a_n(H,t)=0, \tag{3}$$

where the a_i are polynomials in H and t and P is a homogeneous polynomial in H^{2x}, t^y, and F. The only other conditions that $F(H,t)$ must satisfy are that it be real for real H and t and that the appropriate entropy and magnetization per particle be given by

$$S=-(\partial F/\partial T)_H, \qquad M=-(\partial F/\partial H)_T. \tag{4}$$

In order to compute the critical exponents, we expand $F(H,t)$ for fixed t about $H=0$ and for fixed H about $t=0$.

$$F=a(t)+b(t)H+c(t)H^2+\cdots, \tag{5}$$

$$F=d(H)+e(H)t+f(H)t^2+\cdots. \tag{6}$$

When Eqs. (5) and (6) are inserted into the defining Eq. (3) and the coefficients of the various powers of H and t are set equal to zero, the coefficients $a(t)$ to $f(H)$ can be found for any given polynomial P. However, in order to demonstrate the scaling-law equality which comes from Eq. (1), it is only necessary to look at the coefficient of H^2 arising from the term F^n when the expansion given by Eq. (5) is used. This coefficient contains $a^{n-1}(t)\times c(t)$ as well as $a^{n-2}(t)\times b^2(t)$. Since the polynomial, P is homogeneous, these two terms must be of the same order in t. The thermodynamic correspondences are

$$C_{H=0}=-Ta''(t)\sim|t|^{-\alpha'}, \tag{7a}$$

$$M(t)|_{H=0}=-b(t)\sim|t|^{\beta}, \tag{7b}$$

and

$$\chi(t)|_{H=0}=-c(t)\sim|t|^{-\gamma'}, \tag{7c}$$

so that we immediately get

$$(2-\alpha')-\gamma'=2\beta \tag{8}$$

or

$$\alpha'+2\beta+\gamma'=2. \tag{9}$$

This scaling law will be referred to as being of unmixed type since all the critical exponents in it refer to the same variable t. The scaling law of Eq. (2) is consequently of mixed type and will be shown to be weaker in a certain sense than unmixed types. However, another unmixed scaling law immediately presents itself. This is obtained by examining the coefficient of t^2 arising from the term F^n in P when the expansion given by Eq. (6) is used. This coefficient contains $d^{n-1}(H)\times f(H)$ as well as $d^{n-2}(H)\times e^2(H)$. Again, since the polynomial P is homogeneous, these two terms are of the same order. The thermodynamic correspondences are now

$$M(H)|_{t=0}=-d'(H)\sim H^{1/\delta}, \tag{10a}$$

$$S(H)|_{t=0}=-e(H)\sim -H^{\epsilon}, \tag{10b}$$

$$C_H|_{t=0}=-T_cf(H)\sim H^{-\phi}, \tag{10c}$$

[1] M. H. Coopersmith, Phys. Letters **24A**, 700 (1967).
[2] G. S. Rushbrooke, J. Chem. Phys. **39**, 842 (1963).
[3] R. B. Griffiths, Phys. Rev. Letters **14**, 623 (1965).

and the new unmixed scaling law is

$$\phi+2\epsilon-1/\delta=1. \tag{11}$$

Unfortunately, although the exponent δ has already been determined for various ferromagnets,[4] accurate measurements of ϕ and ϵ are not currently available. While ϕ can be measured directly in a manner similar to the measurement of α', ϵ can only be obtained indirectly by measuring $(\partial M/\partial T)_H$ as a function of applied field along the critical isotherm and using the Maxwell relation $(\partial M/\partial T)_H=(\partial S/\partial H)_T$.

It would be of great value to have some experimental verification of Eq. (11), since this equality and the equality of Eq. (1) appear to have a stronger basis than the equality of Eq. (2), that is, they hold for a wider class of functions than Eq. (2). This can be seen from the fact that one can construct algebraic functions for which Eqs. (1) and (11) are satisfied with the equality sign and for which Eq. (2) holds only as an inequality. An example is the algebraic function defined by the (inhomogeneous) polynomial.

$$F^7+2t^3F^5+t^5F^4+t^6F^3+2t^8F^2 +t^2H^4F^2+tH^6F+t^{11}+H^8+H^2t^8=0. \tag{12}$$

This yields the following set of critical exponents.

$$\alpha=\tfrac{1}{3};\quad \alpha'=\tfrac{1}{2};\quad \beta=\tfrac{1}{4};\quad \gamma=\tfrac{4}{3};\quad \gamma'=1;$$
$$\delta=7;\quad \epsilon=\tfrac{2}{7};\quad \phi=\tfrac{4}{7}; \tag{13}$$

with

$$\alpha'+2\beta+\gamma'=2, \tag{14a}$$
$$\phi+2\epsilon-1/\delta=1, \tag{14b}$$
$$\alpha'+\beta(\delta+1)=\tfrac{5}{2}>2. \tag{14c}$$

It is readily seen that one can alter the exponents referring to one variable (within the confines of the appropriate unmixed scaling law) while leaving the exponents referring to the other variable unchanged. For example, a term of the form t^nH^m in the coefficient of a power of F would not change any of the above critical exponents if both n and m were greater than 2. (It would change any critical exponents referring to t and H derivatives of the usual thermodynamic quantities, but these are not being considered at present.) On the other hand, if $n\leq 2$ while $m>2$, the δ, ϵ, ϕ exponents (referring to H) could be changed subject to Eq. (11) while the α, β, γ exponents (referring to t) would remain unchanged. Thus, Eq. (2) is weaker than Eqs. (1) and (11) in the sense that it holds for a smaller class of functions.

The inequality which reduces to the scaling law of Eq. (11) can be derived from thermodynamic considerations in a manner similar to Rushbrooke's derivation of Eq. (1). We start with the identity

$$C_H-C_M=T(\partial M/\partial T)_H^2/\chi, \tag{15}$$

where χ is the isothermal susceptibility. If we now look at the behavior of the right and left hand sides of Eq. (15) as a function of H along the critical isotherm and use the stability condition which goes into Rushbrooke's argument (positivity of C_M), we obtain, after using the above Maxwell relation stated previously, the inequality

$$C_H>T(\partial S/\partial H)_T^2/\chi. \tag{16}$$

This leads to the following inequality when Eqs. (10a)–(10c) are used.

$$-\phi\leq 2(\epsilon-1)-(1/\delta-1) \tag{17}$$

or

$$\phi+2\epsilon-1/\delta\geq 1. \tag{18}$$

A similar inequality can be derived for a fluid in the neighborhood of its critical point but it is less useful experimentally since it involves measuring C_p, $(\partial\rho/\partial T)$, and κ as a function of pressure along the critical isotherm.

ACKNOWLEDGMENT

The author has the pleasure of thanking Professor L. L. Foldy for some very helpful discussions concerning the use of algebraic functions of two variables in the treatment of critical phenomena.

[4] See, for example, the review article by L. P. Kadanoff *et al.*, Rev. Mod. Phys. **39**, 395 (1967).