Existence of a Thermodynamic State with a Broken Geometrical Symmetry in an Electron Gas*

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We consider some possible thermodynamic equilibrium states of a system of electrons with a broken geometrical symmetry. We give the analytic expression for the (quantum-mechanical) static linear-response function of a system of free electrons moving in a uniform magnetic field. For the case where the energy associated with the cyclotron frequency of the system is negligible compared with the Fermi energy, and the temperature of the system is negligible compared with the Fermi temperature, the expression for the static-response function is extremely simple and can be rigorously derived. We analyze the response function and show that it gives various broken-symmetry states in an electron gas.

1. INTRODUCTION

WE study the thermodynamic equilibrium state of an electron gas in a uniform applied magnetic field. We are looking for the possibility of occurrence of a state with broken geometrical symmetry. The basic theorem of our work is introduced in the Sec. 2. The theorem itself is not conceptually new. There have been frequent applications of the idea, in both the relativistic and the nonrelativistic domains, that there exists a new particle or a new mode whenever the denominator of a renormalized propagator vanishes. We introduce a theorem that gives a simple condition, for a system of electrons, for the existence of a thermodynamic equilibrium state with a broken geometrical symmetry. This state will be characterized by a spontaneously induced symmetry-breaking magnetic field in the system. We call the combined system of the electrons and the induced electromagnetic fields the total system.

In Sec. 3, we consider the system of electrons in a normal metal. We give some properties of such a system when it is in the state with a broken geometrical symmetry. In Sec. 4, we apply the result of Sec. 3 to the case of the system of noninteracting electrons. These electrons represent the system of quasielectrons in a metal with a spherical Fermi surface. We give method of calculating the static (bare) linear-response function analytically. The static response function $K(q)$ is given in Eqs. (4.25)–(4.28), which are extremely simple expressions. The expressions are mathematically rigorous under the assumptions that $\omega_c/\epsilon_F \ll 1$, $T/T_F \ll 1$, and $q < p_F$, where ω_c is the cyclotron frequency and ϵ_F, T_F, and p_F are the Fermi energy, the Fermi temperature, and the Fermi momentum, respectively.

Finally, in Sec. 5, we use the results of Sec. 4 to prove that the symmetry-breaking state we are interested in does not occur in the system of quasielectrons when there is no uniform magnetic field present, while it may occur for the system in a uniform magnetic field. We give a qualitative discussion of this latter case. We show, in particular, that the system will spontaneously adopt a state with broken symmetry on the scale of the cyclotron radius, which might be susceptible to laboratory investigation.

2. A THEOREM

We consider a system of N electrons in a large volume V. The interactions of electrons with the electromagnetic fields is governed by the principle of minimal electromagnetic interactions.[1] This principle gives the coupled equations of motion for the electrons and for the electromagnetic fields. The equations of motion for the electromagnetic fields are the Maxwell equations. We consider the case where the electrons are in the nonrelativistic domain. Since we are interested in the thermodynamic equilibrium states of the total system, the electromagnetic fields must be static. Subtracting out the electrostatic self-energy of the electrons in order to have true interactions of the point charges, and taking the thermodynamic limit, we find that the electrons are governed by the Hamiltonian given by[2]

$$H=\sum_\alpha \int d^3x\psi_\alpha{}^\dagger(\mathbf{x})\{(2m)^{-1}[i^{-1}(\partial/\partial\mathbf{x})-(e/c)\mathbf{A}(\mathbf{x})]^2$$
$$+e\phi(\mathbf{x})\}\psi_\alpha(\mathbf{x})+\tfrac{1}{2}\sum_{\alpha,\beta}\iint d^3x_1d^3x_2\psi_\alpha{}^+(\mathbf{x}_1)\psi_\beta{}^+(\mathbf{x}_2)$$
$$\times v(\mathbf{x}_1-\mathbf{x}_2)\psi_\beta(\mathbf{x}_2)\psi_\alpha(\mathbf{x}_1). \quad (2.1)$$

The vector potential $\mathbf{A}(\mathbf{x})$ is governed by the Maxwell equation

$$(\partial^2/\partial\mathbf{x}^2)\mathbf{A}(\mathbf{x})=-(4\pi/c)\mathbf{j}(\mathbf{x}), \quad (2.2)$$

with the gauge given by

$$(\partial/\partial\mathbf{x})\cdot\mathbf{A}(\mathbf{x})=0, \quad (2.3)$$

where the macroscopic current density is given by

$$\mathbf{j}(\mathbf{x})=(ie/2m)\sum_\alpha \lim_{\mathbf{x}'\to\mathbf{x}}[(\partial/\partial\mathbf{x}')-(\partial/\partial\mathbf{x})]$$
$$\times\langle\psi_\alpha{}^\dagger(\mathbf{x}')\psi_\alpha(\mathbf{x})\rangle$$
$$-(e^2/mc)\mathbf{A}(\mathbf{x})\sum_\alpha\langle\psi_\alpha{}^\dagger(\mathbf{x})\psi_\alpha(\mathbf{x})\rangle. \quad (2.4)$$

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[1] See, e.g., S. Gasiorowicz, *Elementary Particle Physics*, (John Wiley & Sons, Inc., New York, 1966); p. 90.

[2] We use the unit $\hbar=1$.

We denote the thermodynamic average $\langle\cdots\rangle$ as

$$\langle\cdots\rangle=\mathrm{Tr}\{\exp[\beta(H-\mu N)]\cdots\}/\mathrm{Tr}\{\exp[\beta(H-\mu N)]\}, \tag{2.5}$$

in which N denotes the number operator and $\beta=1/k_BT$. The field operators $\psi_\alpha(\mathbf{x})$ and $\psi_\alpha{}^\dagger(\mathbf{x})$ satisfy the anticommutation relations. The Hamiltonian, Eq. (2.1), formally contains all Coulomb interactions among electrons, including those which give rise to the familiar plasma modes. In the present case, however, we are not interested in such collective modes, so we will assume that we have carried out proper renormalization so that the electrons described by Eq. (2.1) are quasi-electrons and the two-particle interaction $v(\mathbf{x}-\mathbf{x}')$ is a screened interaction.

The vector potential $\mathbf{A}(\mathbf{x})$, as the solution of Eqs. (2.1) and (2.2), will be the self-consistent field. We set it equal to

$$\mathbf{A}(\mathbf{x})=\mathbf{A}_0(\mathbf{x})+\mathbf{A}_1(\mathbf{x}) \tag{2.6}$$

such that

(a) $\mathbf{A}_0(\mathbf{x})$ is a Maxwell field, that is, $\mathbf{A}_0(\mathbf{x})$ satisfies

$$(\partial^2/\partial\mathbf{x}^2)\mathbf{A}_0(\mathbf{x})=-(4\pi/c)\mathbf{j}^{(0)}(\mathbf{x}), \tag{2.7}$$

with

$$\mathbf{j}^{(0)}(\mathbf{x})=\mathbf{j}(\mathbf{x})\,|_{\mathbf{A}_1\equiv0}, \tag{2.8}$$

and

$$(\partial/\partial\mathbf{x})\cdot\mathbf{A}_0(\mathbf{x})=0; \tag{2.9}$$

(b) $\mathbf{A}_0(\mathbf{x})$ does not break any of the symmetries of the total system. For example, for the case of a uniform external magnetic field, we choose $\mathbf{A}_0(\mathbf{x})$ such that $\mathbf{B}_0(\mathbf{x})=(\partial/\partial\mathbf{x})\times\mathbf{A}_0(\mathbf{x})$ equals the spatial average of $\mathbf{B}(\mathbf{x})=(\partial/\partial\mathbf{x})\times\mathbf{A}(\mathbf{x})$ projected on the direction of the external field. We call the field $\mathbf{A}_1(\mathbf{x})$ the symmetry-breaking field, and the corresponding magnetic field, $\mathbf{B}_1(\mathbf{x})=(\partial/\partial\mathbf{x})\times\mathbf{A}_1(\mathbf{x})$, the symmetry-breaking magnetic field. From Eqs. (2.2), (2.3), (2.6)–(2.9), we then have

$$(\partial^2/\partial\mathbf{x}^2)\mathbf{A}_1(\mathbf{x})=-(4\pi/c)\{\mathbf{j}(\mathbf{x})-\mathbf{j}^{(0)}(\mathbf{x})\}, \tag{2.10}$$

and

$$(\partial/\partial\mathbf{x})\cdot\mathbf{A}_1(\mathbf{x})=0. \tag{2.11}$$

We restrict ourselves to the case in which the symmetry-breaking field $\mathbf{A}_1(\mathbf{x})$ can be treated as a perturbation. We may then assume that $\|\mathbf{A}_1(\mathbf{x})\|$ is sufficiently small to justify use of a linearized equation. We then have

$$\mathbf{j}^{(1)}(\mathbf{x})=-\int d^3x'\mathsf{K}(\mathbf{x}-\mathbf{x}')\mathbf{A}_1(\mathbf{x}'), \tag{2.12}$$

where $\mathbf{j}^{(1)}(\mathbf{x})$ is the current density linear in $\mathbf{A}_1$. The (bare) linear-response tensor $\mathsf{K}(\mathbf{x}-\mathbf{x}')$ is a functional of the field $\mathbf{A}_0(\mathbf{x})$, and the argument $\mathbf{x}-\mathbf{x}'$ is a notation denoting the geometrical symmetry of the unperturbed system. For example, when the unperturbed system has the full translational and rotational symmetry, we have $\mathbf{x}-\mathbf{x}'\equiv|\mathbf{x}-\mathbf{x}'|$, while when the unperturbed system has a translational symmetry in the z direction and translational and rotational symmetries in the x-y plane, then $\mathbf{x}-\mathbf{x}'\equiv(|\mathbf{x}_\perp-\mathbf{x}_\perp'|,\ z-z')$, where $\mathbf{x}_\perp$ denotes the projection of the vector $\mathbf{x}$ on the x-y plane.

For simplicity, we further restrict ourselves to the case where the field $\mathbf{A}_1(\mathbf{x})$ is defined on, and exists in, the subspace in which the unperturbed state of the system has both translational and rotational symmetry. For example, for the latter case shown above, we will be looking for the symmetry-breaking magnetic field $\mathbf{B}_1(\mathbf{x})$ oriented in the z direction and depending on the x and y coordinates only.

We introduce the Fourier transforms

$$\mathbf{j}^{(1)}(\mathbf{x})=\int\frac{d^3q}{(2\pi)^3}\exp(i\mathbf{q}\cdot\mathbf{x})\mathbf{j}^{(1)}(\mathbf{q}),$$

etc. Then, Eqs. (2.10)–(2.12) become

$$\mathbf{q}\cdot\mathbf{A}_1(\mathbf{q})=0, \tag{2.13}$$

$$q^2\mathbf{A}_1(\mathbf{q})=(4\pi/c)\mathbf{j}^{(1)}(\mathbf{q}), \tag{2.14}$$

and

$$j_i^{(1)}(\mathbf{q})=-\sum_j K_{ij}(\mathbf{q})[\mathbf{A}_1(\mathbf{q})]_j, \tag{2.15}$$

where all the vectors and the tensor have components only in the subspace in which the unperturbed state has the full geometrical symmetries. For example, for the case of a uniform external field applied in the z direction, we have $i,j=1,2$ and $\mathbf{q}=(q_x,q_y,0)$. Because of the geometrical symmetries and of the requirement of the gauge invariance, we have[3]

$$K_{ij}(\mathbf{q})=K(q)[\delta_{ij}-(q_iq_j/q^2)], \tag{2.16}$$

where $K(q)$ is a scalar function of $q=(\sum_j q_j^2)^{1/2}$. Substituting Eq. (2.16) into Eq. (2.15) and using Eq. (2.13), we have

$$\mathbf{j}^{(1)}(\mathbf{q})=-K(q)\mathbf{A}_1(\mathbf{q}), \tag{2.17}$$

or, from Eq. (2.14), we have

$$[q^2+(4\pi/c)K(q)]\mathbf{A}_1(\mathbf{q})=0. \tag{2.18}$$

Since one can always transform away the constant vector potential $\mathbf{A}_1(\mathbf{q})\,|_{q=0}$, we can set

$$\mathbf{A}_1(\mathbf{q})\,|_{q=0}\equiv0. \tag{2.19}$$

From Eqs. (2.18) and (2.19), we therefore have the following theorem:

For the linear-response theory with respect to the symmetry-breaking field $\mathbf{A}_1(\mathbf{q})$ that can have components only in the subspace where the unperturbed system has translational and also rotational symmetries, the Maxwell equation gives the following solutions for the thermodynamic equilibrium state of the system:

(1) there always exists the trivial solution

$$\mathbf{A}_1(\mathbf{q})\equiv0,$$

[3] M. R. Schafroth, Helv. Phys. Acta **24**, 645 (1951).

and

(2) the nontrivial solution $\mathbf{A}_1(\mathbf{q})\neq 0$ exists if and only if the relation

$$(4\pi/c)K(q)/q^2=-1 \tag{2.20}$$

is satisfied.

The question as to which of the two solutions is preferred can be answered by finding which has the lowest thermodynamic potential.[4] We consider this in Sec. 3 for a system of electrons in the normal state.

This theorem is not completely new. A similar theorem has been studied by Quinn[5] and others. However, our approach is to solve the equation of motion for the fermions and the Maxwell equations simultaneously. The solutions of these coupled equations will give thermodynamic states of the system. The question of whether such a solution corresponds to a stable or a metastable state will be determined by considering the thermodynamic potential of the total system. Even though the above theorem is based on the linearized equation with respect to the symmetry-breaking field $\mathbf{A}_1(\mathbf{q})$, and so is valid near $\|\mathbf{A}_1(\mathbf{q})\|=0$, it would determine a possible phase boundary. However, to have the properties of the system in a broken-symmetry state, one has to go beyond the linearized theory. In the rest of this paper, we apply this theorem to a metal in the normal conducting state.

3. A SYSTEM OF ELECTRONS IN A NORMAL CONDUCTING STATE

We consider a system of N electrons in a large volume V, moving in a uniform applied magnetic field. We assume that there exists no off-diagonal long-range order[6] in the electron system, so that the system is in the normal state, and ignore the effect of spin. The electrons in a thermodynamic state are governed by the Hamiltonian given by Eq. (2.1), with the one particle scalar potential $\phi(\mathbf{x})\equiv 0$. We assume that the applied magnetic field is in the z direction. Then, the maximum geometrical symmetries that a state of the system can have are the translational symmetry in the z direction and the translational and rotational symmetries in the x-y plane. We might also expect that the thermodynamic equilibrium state of the system will possess these symmetries, just as the ground state of a system, in general, does. We are, however, interested in the occurrence of a state with a lower geometrical symmetry that can arise by inducing a position-dependent, symmetry-breaking magnetic field $\mathbf{B}_1(x,y)$ oriented in the z direction.

According to Eq. (2.6), we separate the field $\mathbf{A}(\mathbf{x})$ as

$$\mathbf{A}(\mathbf{x})=\mathbf{A}_0(\mathbf{x})+\mathbf{A}_1(\mathbf{x}), \tag{3.1}$$

with the field $\mathbf{A}_0(\mathbf{x})$ being chosen as

$$\mathbf{A}_0(\mathbf{x})=\{-B_0y,0,0\}, \tag{3.2}$$

and the symmetry-breaking field $\mathbf{A}_1(\mathbf{x})$ as

$$\mathbf{A}_1(\mathbf{x})=\{A_1^x(x,y),A_1^y(x,y),0\}. \tag{3.3}$$

Then, in the momentum space, Eq. (2.13) gives

$$\mathbf{A}_1(\mathbf{q})=a(q)(q_y\mathbf{i}-q_x\mathbf{j}), \tag{3.4}$$

where $\mathbf{i}$ and $\mathbf{j}$ are the unit vectors along the x and y axes, respectively.

To get the thermodynamic quantities and the properties of the system, we introduce the one-particle-temperature Green's function for the electrons in the system. It is defined as[7]

$$G_{\alpha\beta}(x,x')=-\langle \hat{T}_\tau(\psi_\alpha(x)\bar{\psi}_\beta(x'))\rangle. \tag{3.5}$$

From the properties of a double τ Green's function[7] and our neglect of the influence of spin, we have

$$G_{\alpha\beta}(x,x')=\delta_{\alpha\beta}G(x,x'), \tag{3.6}$$

where

$$G(x,x')=\beta^{-1}\sum_n \exp[-i\omega_n(\tau-\tau')]G_{\omega_n}(\mathbf{x},\mathbf{x}'), \tag{3.7}$$

with $\omega_n=[(2n+1)/\beta]\pi$ $(n=0,\pm1,\pm2,\cdots)$.

We will first consider the case where the system is in the unperturbed state, so that

$$\mathbf{A}_1(\mathbf{x})\equiv 0. \tag{3.8}$$

We denote the one-particle-temperature Green's function in the unperturbed state as $G_{\omega_n}^{(\omega_c)}(\mathbf{x},\mathbf{x}')$, where the index ω_c denotes the cyclotron frequency,

$$\omega_c=eB_0/mc. \tag{3.9}$$

Since the system in the state described by the temperature Green's function $G_{\omega_n}^{(\omega_c)}(\mathbf{x},\mathbf{x}')$ has the maximum geometrical symmetry, one can use this symmetry and the gauge-invariance condition to have [8–10]

$$G_{\omega_n}^{(\omega_c)}(\mathbf{x},\mathbf{x}')=\exp[-i\tfrac{1}{2}(m\omega_c)(y+y')(x-x')]\times\bar{G}_{\omega_n}^{(\omega_c)}(\mathbf{x}-\mathbf{x}'), \tag{3.10}$$

where $\bar{G}_{\omega_n}^{(\omega_c)}(\mathbf{x}-\mathbf{x}')$ denotes a function depending on the argument

$$\mathbf{x}-\mathbf{x}'=(|(\mathbf{x}-\mathbf{x}')_\perp|,z-z'), \tag{3.11}$$

and is obtained by substituting Eq. (3.10) in the equations of motion for the particles and solving them.

We next consider the temperature Green's function $G_{\omega_n}(\mathbf{x},\mathbf{x}')$ for the system in the state with the broken

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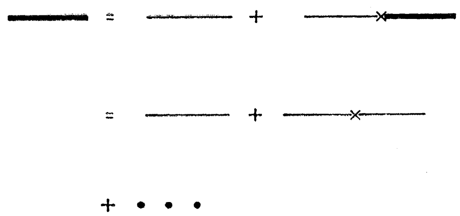

FIG. 1. The diagrammatic expansion of the temperature Green's function with respect to the symmetry breaking field $\mathbf{A}_1(\mathbf{x})$.

geometrical symmetry. Since we have restricted ourselves to the cases where the magnitude of the symmetry-breaking field is sufficiently small so that the field $\mathbf{A}_1(\mathbf{x})$ can be considered as a perturbational field, the perturbed Green's function $G_{\omega_n}(\mathbf{x}, \mathbf{x}')$ can be obtained by using diagrammatic methods.[7] The appropriate diagrams are given in Fig. 1, in which a cross denotes the operator

$$-\frac{e}{mc}\mathbf{A}_1(\mathbf{x})\cdot\left(i^{-1}\frac{\partial}{\partial\mathbf{x}}-\frac{e}{c}\mathbf{A}_0(\mathbf{x})\right)+\frac{e^2}{2mc^2}\mathbf{A}_1^2(\mathbf{x}), \quad (3.12)$$

and the thin lines denote the unperturbed Green's function $G_{\omega_n}{}^{(\omega_c)}(\mathbf{x}, \mathbf{x}')$, while the thick lines denote the perturbed Green's function $G_{\omega_n}(\mathbf{x}, \mathbf{x}')$. Thus, the diagrams in the Fig. 1 give

$$G_{\omega_n}(\mathbf{x}, \mathbf{x}')=G_{\omega_n}{}^{(\omega_c)}(\mathbf{x}, \mathbf{x}')-\frac{e}{mc}\times\int d^3x_1 G_{\omega_n}{}^{(\omega_c)}(\mathbf{x}, \mathbf{x}_1)\{\mathbf{A}_1(\mathbf{x}_1)\cdot[i^{-1}(\partial/\partial\mathbf{x}_1)-(e/c)\mathbf{A}_0(\mathbf{x}_1)]-(e/2c)\mathbf{A}_1^2(\mathbf{x}_1)\}G_{\omega_n}(\mathbf{x}_1, \mathbf{x}') \quad (3.13)$$

or

$$G_{\omega_n}(\mathbf{x}, \mathbf{x}')=G_{\omega_n}{}^{(\omega_c)}(\mathbf{x}, \mathbf{x}')-\frac{e}{mc}\times\int d^3x_1 G_{\omega_n}{}^{(\omega_c)}(\mathbf{x}, \mathbf{x}_1)\mathbf{A}_1(\mathbf{x}_1)\cdot[i^{-1}(\partial/\partial\mathbf{x}_1)-(e/c)\mathbf{A}_0(\mathbf{x}_1)]G_{\omega_n}{}^{(\omega_c)}(\mathbf{x}_1, \mathbf{x}')+O(A_1^2). \quad (3.14)$$

Hence, in principle, one can get all the equilibrium properties as well as the linear-response properties of the system in the state with the broken geometrical symmetry, provided that the symmetry-breaking field $\mathbf{A}_1(\mathbf{x})$ is known.

It is easy to see from Eq. (3.14) that in first-order perturbation theory the chemical potential μ, determined from the equation

$$N/V=(2/\beta)\sum_n \exp(i\omega_n 0+)V^{-1}\int_V d^3x\, G_{\omega_n}(\mathbf{x}, \mathbf{x}), \quad (3.15)$$

will be unchanged. That is,

$$\mu=\lim_{A_1\to 0}\mu+O(A_1^2). \quad (3.16)$$

This can be easily proved by introducing the momentum space representation of the Green's function and considering the parity of the integrand in that space.

Similarly, one can easily prove that

$$\mathbf{j}^{(0)}(\mathbf{x})=\lim_{A_1\to 0}\mathbf{j}(\mathbf{x})=0 \quad (3.17)$$

as one expects from Eqs. (2.7) and (3.2). Note that the current density is given by Eq. (2.4),

$$\mathbf{j}(\mathbf{x})=(ie/m)\beta^{-1}\sum_n \exp(i\omega_n 0+)\times\lim_{\mathbf{x}'\to\mathbf{x}}((\partial/\partial\mathbf{x}')-(\partial/\partial\mathbf{x}))G_{\omega_n}(\mathbf{x}, \mathbf{x}')-(e^2/mc)(2/\beta)\times\sum_n \exp(i\omega_n 0+)G_{\omega_n}(\mathbf{x}, \mathbf{x})\mathbf{A}(\mathbf{x}). \quad (3.18)$$

We now derive an expression for the linear-response function $K(q)$ appearing in Eq. (2.20). Using Eqs. (3.17) and (3.14), we have

$$\mathbf{j}^{(1)}(\mathbf{q})=-(2e^2/m^2c)\beta^{-1}\sum_n \exp(i\omega_n 0+)\times\int\frac{d^3P}{(2\pi)^3}[\tilde{\boldsymbol{\pi}}G_{\omega_n}{}^{(\omega_c)}(\mathbf{p}_+)][\mathbf{A}_1(\mathbf{q})\cdot\tilde{\boldsymbol{\pi}}G_{\omega_n}{}^{(\omega_c)}(\mathbf{p}_-)]-(n_0e^2/mc)\mathbf{A}_1(\mathbf{q}), \quad (3.19)$$

where the operator $\tilde{\boldsymbol{\pi}}$ is defined as

$$\tilde{\boldsymbol{\pi}}=(\tilde{\pi}_x, \tilde{\pi}_y, 0), \quad (3.20)$$

$$\tilde{\pi}_x=p_x-(m\omega_c/2i)\partial/\partial p_y, \quad (3.21)$$

$$\tilde{\pi}_y=p_y+(m\omega_c/2i)\partial/\partial p_x, \quad (3.22)$$

and $\mathbf{p}_\pm$ denotes

$$\mathbf{p}_\pm=\mathbf{p}\pm\tfrac{1}{2}\mathbf{q}.$$

The function $G_{\omega_n}{}^{(\omega_c)}(\mathbf{p})$ is the Fourier transform of the Green's function $\tilde{G}_{\omega_n}{}^{(\omega_c)}(\mathbf{x}-\mathbf{x}')$:

$$\tilde{G}_{\omega_n}{}^{(\omega_c)}(\mathbf{x}-\mathbf{x}')=\int\frac{d^3p}{(2\pi)^3}\exp[i\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}')]G_{\omega_n}{}^{(\omega_c)}(\mathbf{p}), \quad (3.23)$$

and has a symmetry property that

$$G_{\omega_n}{}^{(\omega_c)}(\mathbf{p})=G_{\omega_n}{}^{(\omega_c)}(|\mathbf{p}_\perp|, p_z),$$

where $\mathbf{p}_\perp$ denotes the projection of the vector $\mathbf{p}$ on the p_x-p_y plane. The quantity n_0 appearing in Eq. (3.19) is the number density of the electrons in the unperturbed state;

$$n_0=(2/\beta)\sum_n \exp(i\omega_n 0+)G_{\omega_n}{}^{(\omega_c)}(\mathbf{x}, \mathbf{x}). \quad (3.24)$$

Noting Eqs. (2.15) and (2.16) and considering that

$$\mathbf{q}=(q_x, q_y, 0), \quad (3.25)$$

we have from Eq. (3.19) that

$$K(q)=\mathrm{Tr}\{K_{ij}(\mathbf{q})\}$$
$$=(2e^2/mc)n_0+(2e^2/m^2c)\beta^{-1}\sum_n \exp(i\omega_n 0+)$$
$$\times\int\frac{d^3p}{(2\pi)^3}G_{\omega_n}{}^{(\omega_c)}(\mathbf{p}_+)\,\hat{\pi}^*\cdot\hat{\pi}G_{\omega_n}{}^{(\omega_c)}(\mathbf{p}_-). \quad (3.26)$$

In the above equation, we have a sum in ω_n as well as the integral with respect to $\mathbf{p}$. When the series with respect to ω_n does not converge uniformly, one has to decide the order of the sum and the integral. We will always take the conventional view[7] that the weight function that comes from the statistical distribution in the momentum space should be introduced before the integral with respect to $\mathbf{p}$ is carried out. Since the sum with respect to ω_n gives the statistical distribution, we shall carry out the sum in ω_n before carrying out the integral with respect to $\mathbf{p}$.

It may be shown that, in first-order perturbation theory, the change in the thermodynamic potential for the electrons, Ω_p, cancels that due to the magnetic fields Ω_f, so that the thermodynamic potential of the total system,

$$\Omega=\Omega_p+\Omega_f, \quad (3.27)$$

will not change when the symmetry-breaking field $\mathbf{A}_1(\mathbf{x})$ is switched on. The proof of this statement will be given in Appendix A. Hence, in the linearized theory, any state with broken symmetry has the same thermodynamic energy as the unperturbed state. It should be noted, however, that the linearized theory is valid in the limit where $\|\mathbf{A}_1(\mathbf{q})\|$ tends to zero.

4. APPLICATION TO A METAL AND THE CALCULATION OF THE RESPONSE FUNCTION $K(q)$

We now apply the results of Sec. 3 to a system of noninteracting quasielectrons in a homogeneous and isotropic metal. We thus assume the vanishing two-particle interactions

$$v(\mathbf{x}-\mathbf{x}')\equiv 0. \quad (4.1)$$

In this case, we can easily obtain the unperturbed Green's function $G_{\omega_n}{}^{(\omega_c)}(\mathbf{x},\mathbf{x}')$, given by Eqs. (3.10) and (3.23). The resulting expression for $G_{\omega_n}{}^{(\omega_c)}(\mathbf{p})$ is

$$G_{\omega_n}{}^{(\omega_c)}(\mathbf{p})=2e^{-\eta/2}\sum_{N=0}^{\infty}\frac{(-1)^N L_N(\eta)}{i\omega_n-E_N(p_z)}, \quad (4.2)$$

where $L_N(\eta)$ is the Laguerre polynomial of the order N, and

$$\eta\equiv(2/m\omega_c)(p_x^2+p_y^2), \quad (4.3)$$

$$E_N(p_z)\equiv(1/2m)p_z^2+(N+\tfrac{1}{2})\omega_c-\mu. \quad (4.4)$$

The expression $E_N(p_z)$ in Eq. (4.4) denotes the one-particle energy of the Landau levels[11] measured from the chemical potential μ. An expression similar to Eq. (4.2) was obtained by Bychkov and Gorkov[10] for the double time one-particle Green's function at absolute zero temperature. The unperturbed Green's function is derived in Appendix B. For the electrons in a metal at low temperature, one can set the chemical potential μ approximately equal to the Fermi energy ϵ_F. For convenience, we introduce p_F and v_F through the relations $\mu=(1/2m)p_F^2=\frac{1}{2}p_Fv_F$, so that one can consider the quantities p_F and v_F as the Fermi momentum and the Fermi velocity.

The calculation for the bare linear-response function $K(q)$ given by Eqs. (3.26) and (4.2) is not a trivial problem. We wish to get $K(q)$ in such a simple form that Eq. (2.20) can be solved easily. We here give a method of calculating $K(q)$ analytically.

We note that for a system of electrons with a metallic or higher density, one has two natural expansion parameters, namely, ω_c/μ and $1/\beta\mu\sim T/T_F$, where T_F is the Fermi temperature. We also note that we are not interested in the broken geometrical symmetry within a distance of the order of the lattice constant of a metal. We therefore assume that

$$\omega_c/\mu\ll 1, \quad (4.5)$$

$$1/\beta\mu\ll 1, \quad (4.6)$$

and

$$q<p_F. \quad (4.7)$$

These three are the only assumptions we make in order to get the response function $K(q)$.

We carry out the sum with respect to ω_n in Eq. (3.26) and we define the quantities $D(N,N'\mid p_z)$ and $Q(q)$ as

$$D(N,N'\mid p_z)=\frac{f[E_N(p_z)]-f[E_{N'}(p_z)]}{E_N(p_z)-E_{N'}(p_z)}, \quad (4.8)$$

and

$$Q(q)=\frac{8e^2}{m^2c}\int\frac{d^3p}{(2\pi)^3}\sum_{N,N'}(-1)^{N+N'}D(N,N'\mid p_z)$$
$$\times e^{-\eta'/2}L_{N'}(\eta')[\tfrac{1}{4}q^2+(\mathbf{p}\cdot\mathbf{q})+\tfrac{1}{2}(m\omega_c)$$
$$\times(\eta+4(\partial/\partial\eta)\eta(\partial/\partial\eta))]e^{-\eta/2}L_N(\eta), \quad (4.9)$$

where $f(z)$ is the Fermi distribution function,

$$f(z)=1/[\exp(\beta z)+1]$$

and

$$\eta'=(2/m\omega_c)\,|\,(\mathbf{p}+\mathbf{q})_\perp|^2=\eta+(4/m\omega_c)[(\mathbf{p}\cdot\mathbf{q})+\tfrac{1}{2}q^2].$$

We then have

$$K(q)=Q(q)-Q(0). \quad (4.10)$$

[11] A. B. Pippard, *Reports on Progress in Physics* (The Institute of Physics and The Physical Society, London, 1960), Vol. 23; Proc. Roy. Soc. (London) **A272**, 192 (1963).

In deriving Eq. (4.10), we have noticed that

$$Q(0)=Q(q)\,|_{q=0} = -(2e^2/mc)n_0. \tag{4.11}$$

We give the proof for Eq. (4.11) in Appendix C. We thus need to get $Q(q)$.

One of the difficulties in calculating a physical quantity based on a microscopic theory for an electron gas in a uniform magnetic field is to carry out the sums over the discrete Landau levels. To avoid the difficulty, we introduce the inverse Laplace transform of the Fermi distribution function,[9]

$$f[E_N(p_z)]=(2i)^{-1}\int_{-i\infty+c}^{i\infty+c} dt(\sin\pi t)^{-1}\exp[-E_N(p_z)t]$$

where $0<c<1$. We then have

$$D(N,N'\,|\,p_z)=-\int_{0+}^{\beta-}d\tau(2i)^{-1}\int_{-i\infty+c}^{i\infty+c}dt\,\frac{t}{\sin\pi t} \times\exp[-(\beta-\tau)E_{N'}(p_z)t-\tau E_N(p_z)t]. \tag{4.12}$$

The limiting process at the endpoints of the integral with respect to τ are chosen in such a way that, when (4.12) is substituted in Eq. (4.9), the sums in N and N' are uniformly convergent with respect to the variables τ and t. Noting that $\mathrm{Re}t=c>0$, $0<\tau<\beta$, and $0<\beta-\tau<\beta$, and using the generating function of the Laguerre polynomial,[12]

$$\sum_{N=0}^{\infty}z^N L_N(\eta)=\frac{\exp[\eta z/(1-z)]}{1-z}\quad(|z|<1),$$

we can easily carry out the sums with respect to N and N'.

In order to be able to interchange the integrals with respect to t and τ with the integral with respect to $\mathbf{p}$, we subtract and add the quantity for $\omega_c=0$ in Eq. (4.9). Then, the $\mathbf{p}$ integral can easily be done by using the relations[13]

$$\pi^{-1}\int_0^{\pi}d\theta\cos(z\cos\theta)=J_0(z)$$

and[14]

$$\int_0^{\infty}\exp(-\alpha^2x^2)x^{\nu+1}J_\nu(bx)\,dx=\frac{b^\nu}{(2\alpha^2)^{\nu+1}}\exp\left\{\frac{-b^2}{4\alpha^2}\right\}$$

for $\mathrm{Re}\nu>-1$ and $\mathrm{Re}\alpha^2>0$, where $J_\nu(z)$ is Bessel function. We then have

$$Q(q)-Q_0(q)=\bar{Q}(q)-\lim_{\omega_c\to0}\bar{Q}(q), \tag{4.13}$$

where the function $\bar{Q}(q)$ is defined as

$$\bar{Q}(q)=\int_{0+}^{\beta-}d\tau(2\pi i)^{-1}\int_{-i\infty+c}^{i\infty+c}dtS(t,\tau\,|\,q), \tag{4.14}$$

with

$$S(t,\tau\,|\,q)\equiv-\frac{e^2\omega_c^2}{2c}\left(\frac{m}{2\pi\beta}\right)^{1/2}\frac{t\exp(\beta\mu t)}{t^{1/2}\sin\pi t\sinh\frac{1}{2}\beta\omega_c t}\exp\left\{-\frac{q^2}{m\omega_c}\frac{\sinh\frac{1}{2}(\beta-\tau)\omega_c t\sinh\frac{1}{2}\tau\omega_c t}{\sinh\frac{1}{2}\beta\omega_c t}\right\}$$

$$\times\left\{\frac{\cosh\tau\omega_c t}{\sinh\frac{1}{2}\beta\omega_c t}\left[\frac{\cosh\frac{1}{2}(\beta-\tau)\omega_c t}{\cosh\frac{1}{2}\tau\omega_c t}+\frac{q^2}{m\omega_c}\frac{\sinh^2\frac{1}{2}(\beta-\tau)\omega_c t}{\sinh\frac{1}{2}\beta\omega_c t}\right]-\tanh\tfrac{1}{2}\tau\omega_c t+\frac{q^2}{4m\omega_c}-\frac{q^2}{m\omega_c}\frac{\sinh\frac{1}{2}(\beta-\tau)\omega_c t\cosh\frac{1}{2}\tau\omega_c t}{\sinh\frac{1}{2}\beta\omega_c t}\right\}, \tag{4.15}$$

and the function $Q_0(q)$ is defined as

$$Q_0(q)\equiv Q(q)\,|_{\omega_c=0} =\frac{2e^2}{m^2c}\int\frac{d^3p}{(2\pi)^3}\frac{f(\epsilon_{p+q})-f(\epsilon_p)}{\epsilon_{p+q}-\epsilon_p} \times\{\tfrac{1}{4}q^2+[\mathbf{p}\cdot(\mathbf{p}_\perp+\mathbf{q})]\}, \tag{4.16}$$

where $\epsilon_p=(1/2m)p^2-\mu$. The calculation of Eq. (4.16) is easy for the long-wavelength region, namely,

$$q\ll p_F. \tag{4.17}$$

We have

$$K_0(q)\equiv Q_0(q)-Q_0(0) =\frac{e^2p_F^3}{3\pi^2mc}\left(\frac{q}{2p_F}\right)^2 \times[1-\tfrac{1}{5}(q/2p_F)^2-\tfrac{876}{35}(q/2p_F)^4+O(q/p_F)^6+O(\beta^{-1}\mu^{-1})]. \tag{4.18}$$

$K_0(q)$ is the response function for a system of free electrons without any uniform magnetic field. Note that we have considered the case where $\mathbf{q}=(q_x,q_y,0)$.

We now consider the integral with respect to t. The integrand $S(t,\tau\,|\,q)$ has the following singularities in the t plane:

(a) it has a branch point at $t=0$, so we will introduce a cut in the t plane along the negative real axis;

(b) it has isolated singularities at the zeros of $\sinh\frac{1}{2}\beta\omega_c t$, or at

$$t=t_l =i(2l\pi/\beta\omega_c)\qquad(l=0,\pm1,\pm2,\cdots);$$

(c) it has first-order poles at the zeros of $\cosh\frac{1}{2}\tau\omega_c t$, or at

$$t=t_{n'} =i(2n'+1)\pi/\tau\omega_c\qquad(n'=0,\pm1,\pm2,\cdots)$$

with vanishing residues (which can be easily proved);

[12] *Higher Transcendental Functions*, edited by A. Erdelyi (McGraw-Hill Book Co., New York, 1953), Vol. 2, p. 188–192.
[13] Ref. 12, Chap. 7.
[14] *Tables of Integral Transformations* (McGraw-Hill Book Co., New York, 1954), Vol. 2, p. 29, Eq. (10).

and

(d) it has first-order poles at the zeroes of $\sin\pi t$, or at

$$t=n(n=0, \pm 1, \pm 2, \cdots).$$

Noting the condition Eq. (4.5), $\omega_c \ll \mu$, we have the following boundary properties for the integrand $S(t, \tau \mid q)$:

(i) $$\lim_{\mathrm{Re}\,t\to-\infty, \mathrm{Re}\,t\neq n} S(t, \tau \mid q)=0,$$

and

(ii) $$\lim_{|\mathrm{Im}\,t|\to\infty, \mathrm{Im}\,t\neq t_l, \neq t_n'} S(t, \tau \mid q)=0.$$

Therefore, deforming the path of the integral with respect to t as shown in Fig. 2 and noting boundary properties of $S(t, \tau \mid q)$ given under (i) and (ii), we have

$$\bar{Q}(q)=\bar{Q}^{(0)}(q)+\sum_{l=-\infty}^{\infty}{}' \bar{Q}^{(l)}(q), \tag{4.19}$$

where the prime on the summation sign denotes that $l=0$ is excluded from the sum. Also,

$$\bar{Q}^{(0)}(q)=\int_{0+}^{\beta-} d\tau (2\pi i)^{-1} \int_{c_{-\infty}{}^0} dt S(t, \tau \mid q) \tag{4.20}$$

and

$$\bar{Q}^{(l)}(q)=\int_{0+}^{\beta-} d\tau (2\pi i)^{-1} \int_{c_l} dt S(t, \tau \mid q) \quad (l\neq 0), \tag{4.21}$$

with the contours $c_{-\infty}{}^0$ and $c_l (l\neq 0)$ shown in Fig. 2. The deforming of the path of integration, as shown in Fig. 2 is a standard technique that appears in the theory of the de Haas–van Alphen effect in connection with separating the oscillatory terms with respect to the magnetic field from the nonoscillatory terms. In Eq. (4.19), the function $\bar{Q}^{(0)}(q)$ leads to the nonoscillatory terms while the function

$$\sum_{l=-\infty}^{\infty}{}' \bar{Q}^{(l)}(q)$$

leads to the oscillatory terms.

We now use the assumptions given by Eqs. (4.5)–(4.7). We can, then, easily carry out the integrals in Eqs. (4.20) and (4.21). We have

$$\bar{Q}^{(0)}(q)-\bar{Q}_0{}^{(0)}(q)=-(e^2p_F{}^2/2\pi^2mc)(\omega_c/2\mu)^2 \times[-\tfrac{1}{6}+\tfrac{37}{180}(q/2p_F)^2+O(q/p_F)^4],$$

where

$$\bar{Q}_0{}^{(0)}(q)=\lim_{\omega_c\to 0} \bar{Q}^{(0)}(q)$$

or

$$[\bar{Q}^{(0)}(q)-\bar{Q}_0{}^{(0)}(q)]-[\bar{Q}^{(0)}(0)-\bar{Q}_0{}^{(0)}(0)]$$
$$=-\tfrac{37}{180}(e^2p_F{}^3/2\pi^2mc)(\omega_c/2\mu)^2(q/2p_F)^2[1+O(q/2p_F)^2]. \tag{4.22}$$

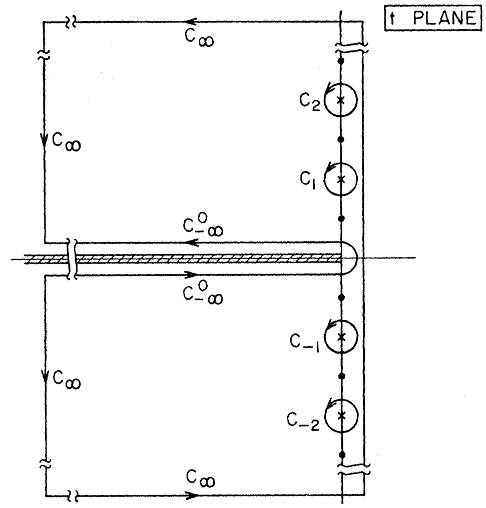

FIG. 2. The contours $c_{-\infty}{}^0$, c_l $(l\neq 0)$, and c_∞ defined in the t plane. The dots on the imaginary axis denote the locations of the poles of the integrand $S(t, \tau|q)$ having vanishing residues.

We also have

$$\sum_{l=-\infty}^{\infty}{}' \bar{Q}^{(l)}(q)=-(e^2p_F{}^3/\pi^2mc)\rho_1(\beta\mu, \omega_c/\mu) \times[J_1(2\mu q/\omega_c p_F)]^2\{1+O(\omega_c/\mu)+O(\beta^{-1}\mu^{-1})\}, \tag{4.23}$$

where the oscillatory function $\rho_1(\beta\mu, \omega_c/\mu)$ is defined as

$$\rho_1(\beta\mu, \omega_c/\mu)\equiv \pi^2/\beta\mu(2\mu/\omega_c)^{1/2} \times\sum_{l=1}^{\infty} \frac{(-1)^l l^{1/2}\cos[2\pi l(\mu/\omega_c)-\frac{1}{4}\pi]}{\sinh(2l\pi^2/\beta\omega_c)}. \tag{4.24}$$

The derivation of Eqs. (4.22) and (4.23) are given in Appendix D, where we also show the relations

$$\lim_{\omega_c\to 0}\sum_{l=-\infty}^{\infty}{}' \bar{Q}^{(l)}(q)=\lim_{q\to 0}\sum_{l=-\infty}^{\infty}{}' \bar{Q}^{(l)}(q)=0.$$

Therefore, combining Eqs. (4.10), (4.13), (4.18) (4.19), (4.22), and (4.23), we have

$$K(q)=K^{(\mathrm{n.o.})}(q)+K^{(\mathrm{osc})}(q), \tag{4.25}$$

with

$$K^{(\mathrm{n.o.})}(q)=K_0(q)-\tfrac{37}{180}(e^2p_F{}^2/2\pi^2mc)(q/2p_F)^2 \times(\omega_c/2\mu)^2[1+O(q/p_F)^2], \tag{4.26}$$

$$K_0(q)=(e^2p_F{}^3/3\pi^2mc)(q/2p_F)^2 \times[1-\tfrac{1}{5}(q/2p_F)^2-\tfrac{876}{35}(q/2p_F)^4+O(\beta^{-1}\mu^{-1})+O(q/p_F)^6], \tag{4.27}$$

and

$$K^{(\mathrm{osc})}(q)=-(e^2p_F{}^3/\pi^2mc)\rho_1(\beta\mu, \omega_c/\mu)(\mu q/\omega_c p_F)^2 \times\left(\frac{J_1(2\mu q/\omega_c p_F)}{\mu q/\omega_c p_F}\right)^2\{1+O(\omega_c/\mu)+O(\beta^{-1}\mu^{-1})\}, \tag{4.28}$$

where the oscillatory function $\rho_1(\beta\mu, \omega_c/\mu)$ is defined in Eq. (4.24). Equation (4.25) gives the analytic expression for the static linear-response function for an electron gas in a uniform magnetic field.

It is interesting to examine the behavior of the various terms in the response function given in the above for the various physical regions one is interested in. We first note that

$$K^{(\mathrm{n.o.})}(q)=K_0(q)[1+O(\omega_c/\mu)^2].$$

That is, the nonoscillatory part of the response function is almost equal to the response function for the system when there is no magnetic field present.

Secondly, we consider the ratio of the oscillatory part to the nonoscillatory part of the response function. The ratio is given by

$$\frac{|K^{(\mathrm{osc})}(q)|}{|K^{(\mathrm{n.o.})}(q)|}=12(\mu/\omega_c)^2\,|\rho_1(\beta\mu, \omega_c/\mu)|\times\left[\frac{J_1(2\mu q/\omega_c p_F)}{\mu q/\omega_c p_F}\right]^2. \quad (4.29)$$

Taking the first term in the expression given by Eq. (4.24) of the oscillatory function $\rho_1(\beta\mu, \omega_c/\mu)$ as is usually done in discussing the de Haas–van Alphen effect[11] in a metal, we can write $\rho_1(\beta\mu, \omega_c/\mu)$ as

$$\rho_1(\beta\mu, \omega_c/\mu)\simeq-(\omega_c/2\mu)^{1/2}\frac{2\pi^2/\beta\omega_c}{\sinh(2\pi^2/\beta\omega_c)}\times\cos(2\pi(\mu/\omega_c)-\tfrac{1}{4}\pi). \quad (4.30)$$

This approximation is valid for the case where $\beta\omega_c\ll1$. Substituting Eq. (4.30) into Eq. (4.28) and inspecting the resulting equation, we can say that, except near the values of μ/ω_c where $\rho_1(\beta\mu, \omega_c/\mu)$ vanishes,

$$|K^{(\mathrm{osc})}(q)|/|K^{(\mathrm{n.o.})}(q)| \begin{cases}\gg1 & \text{if } \beta\omega_c\gg1,\\ \ll1 & \text{if } \beta\omega_c\ll1.\end{cases}$$

for arbitrary q. Thus, one can say that, in the high-temperature and the weak-magnetic-field regime, the nonoscillatory term is dominant, so that one has

$$K(q)\simeq K^{(\mathrm{n.o.})}(q)\simeq K_0(q).$$

On the other hand, in the low-temperature and the strong-magnetic-field regime, the oscillatory term is dominant, so that one can write the response function $K(q)$ as

$$K(q)\simeq K^{(\mathrm{osc})}(q).$$

In Sec. 5, in order to have a rough idea of the behavior of some quantities introduced, we will occasionally use Eq. (4.30) in place of Eq. (4.24). Any result so obtained should not be taken too seriously in the regime where $\beta\omega_c\gg1$.

Finally, for the long-wavelength regime defined by $\mu q/\omega_c p_F\ll1$, we have, for practical purposes,

$$(|K^{(\mathrm{osc})}(q)|/|K^{(\mathrm{n.o.})}(q)|)\gg1,$$

for almost all values of ω_c. Also,

$$K(q)\simeq K^{(\mathrm{osc})}(q)=-(e^2p_F^3/\pi^2mc)\rho_1(\beta\mu, \omega_c/\mu)(\mu q/\omega_c p_F)^2\times\{1-(\mu q/\omega_c p_F)^2+\tfrac{5}{12}(\mu q/\omega_c p_F)^4+O(\mu q/\omega_c p_F)^6\}. \quad (4.31)$$

It is sometimes convenient to set

$$p_F^3/3\pi^2=n_0, \qquad 2\mu q/\omega_c p_F=v_F q/\omega_c, \qquad \omega_P^2=4\pi n_0 e^2/m,$$

where ω_P denotes the plasma frequency.

5. ANALYSIS OF A BROKEN-SYMMETRY STATE FOR A SYSTEM OF NONINTERACTING ELECTRONS

In this section, we make use of the results of Sec. 4 and analyze them. We first prove that in the weak field and/or the high-temperature regime (that is, $\beta\omega_c\ll1$), one cannot have a state with broken symmetry for a system of noninteracting electrons. To prove this, we substitute Eq. (4.26) into Eq. (2.20) and obtain

$$\tfrac{1}{4}(\omega_P^2/c^2p_F^2)\{1-\tfrac{1}{5}(q/2p_F)^2-\tfrac{876}{35}(q/2p_F)^4+O(q/p_F)^6+O(\omega_c/\mu)^2+O(\beta^{-1}\mu^{-1})^2\}=-1. \quad (5.1)$$

We can immediately see that Eq. (5.1) does not have a solution for a metal subject to Eqs. (4.5)–(4.7). Thus the statement is proved. The case when the homogeneous magnetic field is zero is contained in the above statement.

Since the nonoscillatory part $K^{(\mathrm{n.o.})}(q)$ does not give a state with a broken symmetry, it should arise from the oscillatory part $K^{(\mathrm{osc})}(q)$ if at all. We will thus consider the regime where we have

$$12(\mu/\omega_c)^2\,|\rho_1(\beta\mu, \omega_c/\mu)|\left(\frac{J_1(2\mu q/\omega_c p_F)}{\mu q/\omega_c p_F}\right)^2\gg1, \quad (5.2)$$

so that we have

$$|K^{(\mathrm{osc})}(q)|/|K^{(\mathrm{n.o.})}(q)|\gg1 \quad (5.3)$$

or

$$K(q)=K^{(\mathrm{osc})}(q). \quad (5.4)$$

Later, we will prove that this condition is always satisfied by any broken-symmetry solution we may get.

In order to see what we have for a long-wavelength limit (that is, $\mu q/\omega_c p_F\ll1$), we consider only the first two terms in the right side of Eq. (4.31) and substitute them into Eq. (2.20). We then have

$$q^2=(2\omega_c/v_F)^2\{1-\tfrac{4}{3}(c\omega_c/v_F\omega_P)^2[\rho_1(\beta\mu, \omega_c/\mu)]^{-1}\}. \quad (5.5)$$

This solution was independently given by Quinn.[5]

We next consider the general case. We substitute Eqs. (5.4) and (4.28) into Eq. (2.20) and obtain

$$\mathrm{sign}[\rho_1(\beta\mu, \omega_c/\mu)][J_1(x)/Dx]^2=1, \quad (5.6)$$

where x is defined as

$$x \equiv 2\mu q/\omega_c p_F, \tag{5.7}$$

and D is defined as

$$D \equiv \frac{(c/v_F)(\mu/\omega_P)}{(3(\mu/\omega_c)^2 \,|\, \rho_1(\beta\mu, \omega_c/\mu)\,|)^{1/2}}. \tag{5.8}$$

Thus, we have the following theorem:

Equation (2.20) has solutions if and only if the following two conditions are satisfied:

(a) $$\rho_1(\beta\mu, \omega_c/\mu) > 0 \tag{5.9}$$

or, when Eq. (4.30) is valid, Eq. (5.9) is satisfied if

$$\tfrac{3}{8} < \mu/\omega_c - [\mu/\omega_c] < \tfrac{7}{8},$$

where $[\mu/\omega_c]$ is the largest integer not exceeding the value of μ/ω_c, and

(b) $$J_1(x) = \pm Dx. \tag{5.10}$$

The second condition, Eq. (5.10), is the condition of having simultaneous solutions of the equations

$$y = J_1(x) \tag{5.11}$$

and two linear equations

$$y = Dx \tag{5.12}$$

or

$$y = -Dx. \tag{5.13}$$

This can easily be satisfied by drawing corresponding curves in the x-y plane and by reading the x coordinates of the intersecting points, as shown in the Fig. 3. The magnitude of the slopes for the two linear equations are equal and are given by D. The parameter D is a function of the temperature, the uniform magnetic field, and the Fermi energy, and can be obtained from the defining Eq. (5.8).

We can prove that Eq. (5.10) is consistent with Eq. (5.3). To see this, we rewrite Eq. (4.29) as

$$|\, K^{(\mathrm{osc})}(q) \,|/|\, K^{(\mathrm{n.o.})}(q) \,| = (4c\mu/v_F\omega_P)^2 (J_1(x)/Dx)^2. \tag{5.14}$$

Substituting Eq. (5.10) into Eq. (5.14), we have

$$|\, K^{(\mathrm{osc})}(q) \,|/|\, K^{(\mathrm{n.o.})}(q) \,| = (4c\mu/v_F\omega_P)^2 \gg 1. \tag{5.15}$$

We have thus proved the validity of using the relation

$$K(q) \simeq K^{(\mathrm{osc})}(q)$$

in getting the broken-symmetry solutions.

We now consider the solutions of Eq. (5.10) as a function of the parameter D. Since one has[13]

$$\lim_{x\to 0} (\partial/\partial x) J_1(x) = \tfrac{1}{2},$$

it is necessary to have

$$D < \tfrac{1}{2} \tag{5.16}$$

in order to have a solution. From Fig. 3, one can see

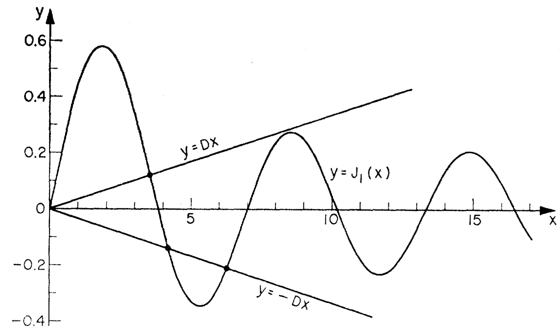

FIG. 3. The curves $y = J_1(x)$ and $y = \pm Dx$. The dots indicate the solutions of Eq. (5.10).

that when the value of D decreases from the critical value of $\frac{1}{2}$ there exists only one solution of Eq. (5.10) until the line $y = -Dx$ touches the curve $y = J_1(x)$. This occurs at

$$D \simeq 0.06, \tag{5.17}$$

and the solution is located in the region where

$$q/p_F < 5.2(\omega_c/\mu). \tag{5.18}$$

When the parameter D is further decreased, the curve $y = Dx$ will touch the curve $y = J_1(x)$ at

$$D \simeq 0.03. \tag{5.19}$$

And in the region

$$0.03 < D < 0.06$$

we have three solutions for Eq. (5.10). Out of three solutions, two solutions come from the intersections of the line $y = -Dx$ with the curve $y = J_1(x)$ and the remaining one comes from the intersection of the line $y = Dx$ with the curve $y = J_1(x)$. All three solutions are located in the region

$$q/p_F < 8.4(\omega_c/\mu). \tag{5.20}$$

To study the dependence of the solutions on ω_c/μ qualitatively, we set

$$D_0 \equiv |\cos[2\pi(\mu/\omega_c) - \tfrac{1}{4}\pi]|^{1/2} D, \tag{5.21}$$

so that, by considering Eqs. (5.8) and (4.30), we have

$$D \geqq D_0. \tag{5.22}$$

Thus, starting from $D = D_0$ (or setting $|\cos[2\pi(\mu/\omega_c) - \frac{1}{4}\pi]| = 1$), we see that D increases as $|\cos[2\pi(\mu/\omega_c) - \frac{1}{4}\pi]|$ decreases. So, the solutions of Eq. (5.6) depend sensitively on $|\cos[2\pi(\mu/\omega_c) - \frac{1}{4}\pi]|$. In Fig. 3, when $|\cos[2\pi(\mu/\omega_c) - \frac{1}{4}\pi]|$ decreases, the lines $y = \pm Dx$ are rotated away from the x axis providing various solutions of Eq. (5.10) until D becomes 0.5.

We shall finally study the effects of the temperature on the solutions of Eq. (5.10). From Eqs. (5.8), (4.30), and (5.21), we have

$$\frac{\sinh(2\pi^2/\beta\omega_c)}{2\pi^2/\beta\omega_c} \simeq \tfrac{3}{2}\sqrt{2}(\mu/\omega_c)^{3/2} (v_F\omega_P/c\mu)^2 D_0^2. \tag{5.23}$$

This equation can also be solved easily by considering the right side as a known parameter and using the graphical method as we have done in solving Eq. (5.10). Since we have

$$\lim_{x\to 0} (d/dx) \sinh x = 1,$$

we should have

$$D_0 > (\tfrac{1}{3}\sqrt{2})^{1/2}(\omega_c/\mu)^{3/4}(c\mu/v_F\omega_P), \qquad (5.24)$$

in order that Eq. (5.23) could have a solution. We consider the case when $\omega_c/\mu \sim 10^{-4}$ and $(c/v_F)(\mu/\omega_P) \sim 10^2$, which are typical values for these parameters. We then have

$$\frac{\sinh(2\pi^2/\beta\omega_c)}{2\pi^2/\beta\omega_c} \sim 10^2 D_0^2.$$

So we can have $D_0 \sim 10^{-1}$ for $\beta\omega_c \sim 2\pi^2$ and, as we can see from Fig. 3, we can easily have a solution with a wave vector q of the order of $\omega_c/\mu p_F$, for the case we are considering, with temperature in the region where $\beta\omega_c \sim 2\pi^2$. A full quantitative analysis of Eqs. (5.6) and (5.8) will be made in a future publication.

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APPENDIX A

Using the standard technique to get the thermodynamic potential,[7] we have, for the thermodynamic potential Ω_p of the electron system,

$$\Omega_p = \Omega_p(\lambda)\,|_{\lambda=1}, \qquad (A1)$$

$$\Omega_p{}^0 = \lim_{A_1\to 0} \Omega_p, = \Omega_p(\lambda)\,|_{\lambda=0}, \qquad (A2)$$

with

$$\frac{\partial}{\partial\lambda}\Omega_p(\lambda) = \frac{2}{\beta}\sum_n \exp(i\omega_n 0+)\int d^3x \lim_{\mathbf{x}'\to\mathbf{x}} \{[-(e/mc)\mathbf{A}_1(\mathbf{x})\cdot(i^{-1}(\partial/\partial\mathbf{x})-(e/c)\mathbf{A}_0(\mathbf{x})) + (e^2\lambda/mc^2)\mathbf{A}_1{}^2(\mathbf{x})][G_{\omega_n}(\mathbf{x},\mathbf{x}')]_{A_1\to\lambda A_1}\}$$

$$= \frac{2}{\beta}\sum_n \exp(i\omega_n 0+)\int d^3x \lim_{\mathbf{x}'\to\mathbf{x}} \{-(e/mc)\mathbf{A}_1(\mathbf{x})\cdot(i^{-1}(\partial/\partial\mathbf{x})-(e/c)\mathbf{A}_0(\mathbf{x}))G_{\omega_n}{}^{(\omega_c)}(\mathbf{x},\mathbf{x}')\}$$

$$+\frac{e^2\lambda}{mc^2}n_0\int d^3x\mathbf{A}_1{}^2(\mathbf{x}) + \frac{2}{\beta}\sum_n \exp(i\omega_n 0+)\int d^3x \lim_{\mathbf{x}'\to\mathbf{x}} \left\{-\frac{e}{mc}\mathbf{A}_1(\mathbf{x})\cdot\left(i^{-1}\frac{\partial}{\partial\mathbf{x}}-\frac{e}{c}\mathbf{A}_0(\mathbf{x})\right)\left[-\frac{\lambda e}{mc}\int d^3x_1 G_{\omega_n}{}^{(\omega_c)}(\mathbf{x},\mathbf{x}_1)\right.\right.$$

$$\left.\left.\times\mathbf{A}_1(\mathbf{x}_1)\cdot\left(i^{-1}\frac{\partial}{\partial\mathbf{x}_1}-\frac{e}{c}\mathbf{A}_0(\mathbf{x}_1)\right)G_{\omega_n}{}^{(\omega_c)}(\mathbf{x}_1,\mathbf{x}')\right]\right\}+O(A_1{}^3). \qquad (A3)$$

We can, however, easily see that

$$\int d^3x \lim_{\mathbf{x}'\to\mathbf{x}} \mathbf{A}_1(\mathbf{x})\cdot(i^{-1}(\partial/\partial\mathbf{x})-(e/c)\mathbf{A}_0(\mathbf{x}))G_{\omega_n}{}^{(\omega_c)}(\mathbf{x},\mathbf{x}') = [\mathbf{A}_1(q)]_{q=0}\cdot\int\frac{d^3p}{(2\pi)^3}\,\boldsymbol{\pi}G_{\omega_n}{}^{(\omega_c)}(\mathbf{p}) = 0, \qquad (A4)$$

and

$$\int d^3x \lim_{\mathbf{x}'\to\mathbf{x}} \left\{-\frac{e}{mc}\mathbf{A}_1(\mathbf{x})\cdot\left(i^{-1}\frac{\partial}{\partial\mathbf{x}}-\frac{e}{c}\mathbf{A}_0(\mathbf{x})\right)\left[-\frac{\lambda e}{mc}\int d^3x_1 G_{\omega_n}{}^{(\omega_c)}(\mathbf{x},\mathbf{x}_1)\mathbf{A}_1(\mathbf{x}_1)\cdot\left(i^{-1}\frac{\partial}{\partial\mathbf{x}_1}-\frac{e}{c}\mathbf{A}_0(\mathbf{x}_1)\right)G_{\omega_n}{}^{(\omega_c)}(\mathbf{x}_1,\mathbf{x}')\right]\right\}$$

$$= \frac{e\lambda}{imc}\int d^3x \lim_{\mathbf{x}'\to\mathbf{x}} \mathbf{A}_1(\mathbf{x})\left\{\left(\frac{\partial}{\partial\mathbf{x}'}-\frac{\partial}{\partial\mathbf{x}}\right)\exp[i\tfrac{1}{2}(m\omega_c)(y+y')(x-x')]\left[-\frac{e}{mc}\int d^3x_1 G_{\omega_n}{}^{(\omega_c)}(\mathbf{x},\mathbf{x}_1)\mathbf{A}_1(\mathbf{x})\right.\right.$$

$$\left.\left.\cdot\left(i^{-1}\frac{\partial}{\partial\mathbf{x}_1}-\frac{e}{c}\mathbf{A}_0(\mathbf{x}_1)\right)G_{\omega_n}{}^{(\omega_c)}(\mathbf{x}_1,\mathbf{x}')\right]\right\}. \qquad (A5)$$

Substituting Eqs. (A4) and (A5) into Eq. (A3), and noting Eq. (3.19), we have

$$\Omega_p - \Omega_p{}^0 = -(2c)^{-1}\int\frac{d^3q}{(2\pi)^3}\,\mathbf{j}^{(1)}(\mathbf{q})\cdot\mathbf{A}_1(-\mathbf{q}) + O(A_1{}^3). \qquad (A6)$$

Substituting Eq. (2.17) into Eq. (A6) and using the condition (2.20) on the existence of a nontrivial solution $\mathbf{A}_1(\mathbf{q})$, we have

$$\Omega_p - \Omega_p{}^0 = -\int d^3x\frac{\mathbf{B}_1(\mathbf{x})^2}{8\pi} + O(B_1{}^3). \qquad (A7)$$

This equation gives the changes in the thermodynamic potential of the particle system when the symmetry-breaking field $\mathbf{A}_1(\mathbf{q})$, satisfying the Maxwell equation, is switched on. However, the first term on the right side of Eq. (A7) cancels the increase in the field energy due to the magnetic field. Therefore, the total increase of the thermodynamic potential of the total system is

$$\Omega-\Omega_0=O(A_1{}^3), \tag{A8}$$

where

$$\Omega_0=\lim_{A_1\to 0}\Omega.$$

This gives the proof for the statement that the thermodynamic potential does not change in first-order theory.

APPENDIX B

We here give a derivation of the temperature Green's function for a system of free electrons in a uniform magnetic field. The equations of motion are given by

$$\{i\omega_n-[(2m)^{-1}(i^{-1}(\partial/\partial\mathbf{x})-(e/c)\mathbf{A}_0(\mathbf{x}))^2-\mu]\}\times G_{\omega_n}{}^{(\omega_c)}(\mathbf{x},\mathbf{x}')=\delta^3(\mathbf{x}-\mathbf{x}') \tag{B1}$$

and

$$\{i\omega_n-[(2m)^{-1}(i^{-1}(\partial/\partial\mathbf{x}')+(e/c)\mathbf{A}_0(\mathbf{x}'))^2-\mu]\}\times G_{\omega_n}{}^{(\omega_c)}(\mathbf{x},\mathbf{x}')=\delta^3(\mathbf{x}-\mathbf{x}'), \tag{B2}$$

with the vector potential $\mathbf{A}_0(\mathbf{x})$ chosen as

$$\mathbf{A}_0(\mathbf{x})=\{-B_0y,0,0\}. \tag{B3}$$

We set

$$G_{\omega_n}{}^{(\omega_c)}(\mathbf{x},\mathbf{x}')=\exp[-i\tfrac{1}{2}(m\omega_c)(y+y')(x-x')]\times\bar{G}_{\omega_n}{}^{(\omega_c)}(\mathbf{x}-\mathbf{x}') \tag{B4}$$

and

$$\bar{G}_{\omega_n}{}^{(\omega_c)}(\mathbf{x}-\mathbf{x}')=\bar{G}_{\omega_n}{}^{(\omega_c)}(|\mathbf{x}_\perp-\mathbf{x}_\perp'|,z-z') \tag{B5}$$

as by Eqs. (3.10) and (3.11). Substituting Eq. (B4) into Eqs. (B1) and (B2), we have

$$\{i\omega_n-[\tfrac{1}{2}\omega_c(\xi^2+\hat{p}_\xi{}^2)+(2m)^{-1}(i^{-1}(\partial/\partial r_z))^2-\mu]\}\times\bar{G}_{\omega_n}{}^{(\omega_c)}(\mathbf{r})=\delta^3(\mathbf{r}) \tag{B6}$$

and

$$\{i\omega_n-[\tfrac{1}{2}\omega_c(\hat{\eta}^2+\hat{p}_\eta{}^2)+(2m)^{-1}(i^{-1}(\partial/r_z))^2-\mu]\}\times\bar{G}_{\omega_n}{}^{(\omega_c)}(\mathbf{r})=\delta^3(\mathbf{r}). \tag{B7}$$

We have defined the relative vector $\mathbf{r}=(r_x,r_y,r_z)$ as

$$\mathbf{r}=\mathbf{x}-\mathbf{x}' \tag{B8}$$

and introduced the operators $\hat{\eta}$, $\hat{p}_\eta$, $\hat{\xi}$, and $\hat{p}_\xi$ as

$$\hat{\eta}\equiv-(m\omega_c)^{-1/2}(i^{-1}(\partial/\partial r_y)+\tfrac{1}{2}(m\omega_c)r_x), \tag{B9}$$

$$\hat{p}_\eta\equiv-(m\omega_c)^{-1/2}(i^{-1}(\partial/\partial r_x)-\tfrac{1}{2}(m\omega_c)r_y), \tag{B10}$$

$$\hat{\xi}\equiv(m\omega_c)^{-1/2}(i^{-1}(\partial/\partial r_x)+\tfrac{1}{2}(m\omega_c)r_y), \tag{B11}$$

$$\hat{p}_\xi\equiv(m\omega_c)^{-1/2}(i^{-1}(\partial/\partial r_y)-\tfrac{1}{2}(m\omega_c)r_x). \tag{B12}$$

Adding and subtracting Eq. (B7) from Eq. (B6), we have

$$\{i\omega_n-(2m)^{-1}[(i^{-1}(\partial/\partial\mathbf{r})^2+\tfrac{1}{2}(m\omega_c)r_\perp{}^2]+\mu\}\times\bar{G}_{\omega_n}{}^{(\omega_c)}(\mathbf{r})=\delta^3(\mathbf{r}) \tag{B13}$$

and

$$(r_y(\partial/\partial r_x)-r_x(\partial/\partial r_y))\bar{G}_{\omega_n}{}^{(\omega_c)}(\mathbf{r})=0, \tag{B14}$$

where $r_\perp$ is defined as

$$r_\perp\equiv r_x{}^2+r_y{}^2. \tag{B15}$$

Eq. (B14) will give

$$\bar{G}_{\omega_n}{}^{(\omega_c)}(\mathbf{r})=\bar{G}_{\omega_n}{}^{(\omega_c)}(r_\perp,r_z), \tag{B16}$$

which is just Eq. (B5). Introducing the following expansion:

$$\bar{G}_{\omega_n}{}^{(\omega_c)}(r_\perp,r_z)=\int_{-\infty}^{\infty}\frac{dp_z}{2\pi}\exp(ip_zr_z)\sum_{N=0}^{\infty}a_N(p_z)\times\exp(-\tfrac{1}{2}\rho^2)L_N(\rho^2), \tag{B17}$$

where $L_N(\rho^2)$ is the Laguerre polynomial of the Nth order and ρ is defined as

$$\rho^2=\tfrac{1}{2}(m\omega_c)r_\perp{}^2, \tag{B18}$$

and substituting it in Eq. (B13), we have

$$a_N(p_z)=m\omega_c/2\pi[i\omega_n-E_N(p_z)]^{-1}, \tag{B19}$$

where $E_N(p_z)$ is given by

$$E_N(p_z)=(1/2m)p_z{}^2+(N+\tfrac{1}{2})\omega_c-\mu. \tag{B20}$$

Therefore, we have

$$\bar{G}_{\omega_n}{}^{(\omega_c)}(\mathbf{x}-\mathbf{x}')=\frac{m\omega_c}{2\pi}\int_{-\infty}^{\infty}\frac{dp_z}{2\pi}\sum_{N=0}^{\infty}e^{-\rho^2/2}L_N(\rho^2)\times\frac{\exp[ip_z(z-z')]}{i\omega_n-E_N(p_z)}. \tag{B21}$$

Taking the Fourier transform and using the formula[15]

$$\int_0^\infty d\rho\rho^{1/2}e^{-\rho^2/2}L_N(\rho^2)J_0(\rho\eta^{1/2})(\rho\eta^{1/2})^{1/2}=(-1)^Ne^{-\eta/2}\eta^{1/4}L_N(\eta)\qquad(\eta>0),$$

where η is given by Eq. (4.3), we can easily get Eq. (4.2).

APPENDIX C

We here give the proof of Eq. (4.11). From Eq. (4.9) we have

$$Q(q)\,|_{q=0}=\frac{e^2\omega_c{}^2}{\pi c}\int_{-\infty}^{\infty}\frac{dp_z}{2\pi}\times\sum_{N=0}^{\infty}\{(N+1)D(N,N+1\mid p_z)+ND(N,N-1\mid p_z)\}$$

$$=-\frac{2e^2}{mc}\frac{m\omega_c}{\pi}\int_{-\infty}^{\infty}\frac{dp_z}{2\pi}\sum_{N=0}^{\infty}f(E_N(p_z)). \tag{C1}$$

[15] Ref. 14; p. 13, Eq. (3).

We have used the fact that the set of functions $\{e^{-\eta/2}L_N(\eta)\}$ is complete. From Eq. (3.24), we have

$$n_0=(2/\beta)\sum_n \exp(i\omega_n 0+)G_{\omega_n}{}^{(\omega_c)}(\mathbf{x},\mathbf{x})$$

$$=\frac{m\omega_c}{\pi}\int_{-\infty}^{\infty}\frac{dp_z}{2\pi}\sum_{N=0}^{\infty}f(E_N(p_z)) \quad \text{(C2)}$$

by using Eq. (B21). Substituting Eq. (C2) into Eq. (C1) we get Eq. (4.11).

APPENDIX D

In this Appendix, we carry out the integrals with respect to variables t and τ appears in Eqs. (4.20) and (4.21). Using assumptions given by Eqs. (4.5)–(4.7), we can easily carry out the t integrals by using the method similar to that of Luttinger.[9]

We will calculate the nonoscillatory term $\bar{Q}^{(0)}(q)$ first. We note that, on the contours of integration $c_{-\infty}{}^0$, we have

$$-\infty\leq \mathrm{Re}\,t\leq 0.$$

We therefore set

$$\xi=\beta\mu t. \quad \text{(D1)}$$

Then, due to the factor e^{ξ} in the integrand $S(\xi/\beta\mu,\tau\,|\,q)$, the contributions from the large values of $|\xi|$ to the integral are vanishingly small. Thus it is natural to expand the integrand $S(\xi/\beta\mu,\tau\,|\,q)$ in the powers of $\omega_c\xi/\mu$ and $\xi/\beta\mu$. We then have, after some algebra, Eq. (4.22).

We next consider the oscillatory term $\bar{Q}^{(l)}(q)$. We set

$$\xi=\beta\mu(t-t_l). \quad \text{(D2)}$$

We have

$$\bar{Q}^{(l)}(q)=\int_{0+}^{\beta-}d\tau(2\pi i)^{-1}\int_{\Gamma_0}\frac{d\xi}{\beta\mu}S\left(t_l+\frac{\xi}{\beta\mu},\tau\,\Big|\,q\right), \quad (l\neq 0) \quad \text{(D3)}$$

where the contour Γ_0 is that encircling the origin of the ξ plane counterclockwise. Since, for $l\neq 0$, the origin of the ξ plane is an isolated singularity and one can deform the contour Γ_0 in an arbitrary ϵ neighborhood of the origin, one can consider $|\xi|$ arbitrarily small on the contour Γ_0. We can thus again introduce an expansion in the powers of $\omega_c\xi/\mu$ and $\xi/\beta\mu$. This is again a standard method of calculating the leading terms for a physical quantity in connection with the de Haas–van Alphen oscillations.[9] We then have, after some algebra,[13]

$$\bar{Q}^{(l)}(q)=-\frac{e^2p_F{}^2}{mc}\left(\frac{m}{2\pi\beta}\right)^{1/2}\frac{(-1)^l t_l\exp(\beta\mu t_l)}{t_l{}^{1/2}\sin\pi t_l}$$
$$\times\int_0^{\pi}\frac{d\theta}{\pi}(\cos 2\theta)J_0\left(\frac{4\mu q}{\omega_c p_F}\sin\theta\right)\{1+O(\omega_c/\mu)+O(\beta\mu^{-1})\} \quad (l\neq 0)$$
$$=-\frac{e^2p_F{}^2}{mc}\left(\frac{m}{2\pi\beta}\right)^{1/2}\frac{(-1)^l t_l\exp(\beta\mu t_l)}{t_l{}^{1/2}\sin\pi t_l}$$
$$\times\left\{J_1\left(\frac{2\mu q}{\omega_c p_F}\right)\right\}^2\{1+O(\omega_c/\mu)+O(\beta^{-1}\mu^{-1})\}. \quad \text{(D4)}$$

In deriving Eq. (D4), we have used the Schläflis integral representation for Bessel functions[16] which reduces, for Bessel functions of an integer order, to

$$J_n(z)=(2\pi i)^{-1}(\tfrac{1}{2}z)^n\int_{\Gamma_0}d\xi\,\frac{\exp[\xi-(\frac{1}{2}z)^2\xi^{-1}]}{\xi^{n+1}}$$

where the contour Γ_0 is that appearing in Eq. (D3).

We shall note the following properties of the function $\bar{Q}^{(l)}(q)$ $(l\neq 0)$:

(a) $$\lim_{\omega_c\to 0}\bar{Q}^{(l)}(q)=0, \quad (l\neq 0)$$

since the dominant factor in $\bar{Q}^{(l)}(q)$ when $\omega_c\to 0$ is

$$(\sin\pi t_l)^{-1}=[i\sinh(2l\pi^2/\beta\omega_c)]^{-1},$$

(b) $$\lim_{q\to 0}\bar{Q}^{(l)}(q)=0,$$

and

(c) $$\sum_{l=-\infty}^{\infty}{}'\frac{(-1)^l t_l\exp(\beta\mu t_l)}{t_l{}^{1/2}\sin\pi t_l}=2\left(\frac{2\pi}{\beta\omega_c}\right)^{1/2}\sum_{l=1}^{\infty}\frac{(-1)^l l^{1/2}\cos(2\pi l(\mu/\omega_c)-\frac{1}{4}\pi)}{\sinh(2l\pi^2/\beta\omega_c)}.$$

We then have Eq. (4.23) and the relations

$$\lim_{\omega_c\to 0}\sum_{l=-\infty}^{\infty}{}'\bar{Q}^{(l)}(q)=\lim_{q\to 0}\sum_{l=-\infty}^{\infty}{}'\bar{Q}^{(l)}(q)=0. \quad \text{(D5)}$$

[16] H. Jeffreys and B. S. Jeffreys, *Method of Mathematical Physics*, (Cambridge University Press, New York, 1959), p. 575.