Type 7. For spin $\frac{1}{2}$ one may take an isospinor N which by parity transforms as

$$N \rightarrow \pm iN\gamma_0, \tag{6.64}$$

and one should add an interaction which restricts the internal symmetry group to be just $SU(2)$. Then, in the basis $\psi = (N, N^\dagger \epsilon)$,

$$D^u(g) = \begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix}, \quad D^u(\mathcal{P}_0) = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}, \quad D^u(\Theta_0) = \begin{pmatrix} 0 & -\epsilon \\ \epsilon & 0 \end{pmatrix}. \tag{6.65}$$

A Lagrangian which achieves the above can be easily constructed following a method similar to that used for type 3. Similarly, for spin 1, one may use a vector K_μ which is an isospinor and which by parity transforms as

$$K_\mu \rightarrow iK_\mu \epsilon_\mu. \tag{6.66}$$

In the basis $V_\mu = (K_\mu, K_\mu{}^\dagger \epsilon)$ the matrices D^u are again given by (6.65).

The basis fields for the examples discussed above for Table IV fit into general forms analogous to those described after Eq. (6.33) for the types of Table II. These general forms are collected in Table VIII, which is a sort of summary of the results of this section.

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Partial-Wave Analysis in Terms of the Homogeneous Lorentz Group

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The aim of this paper is to generalize Toller's work on elastic forward scattering and to expand the general two-body amplitude for all values of momentum transfer in terms of unitary representations of $SO(3,1)$. The authors' concern is covariant inclusion of spin.

1. INTRODUCTION

IN a series of recent papers, Toller[1] has made a fundamental advance in noticing and exploiting the extra $O(3,1)$ invariance possessed by the elastic forward-scattering amplitude. The new invariance leads him to an expansion of the amplitude in terms of unitary representations of the group $SO(3,1)$. This is in contrast to the normal partial-wave analysis which is an expansion in terms of unitary representations of $SO(3)$—a much smaller structure. The new expansion—embodying the higher symmetry—leads to newer insights; for example, if the new partial-wave amplitude, labeled with the four-dimensional generalized angular momentum σ, is assumed to be meromorphic for complex σ, one finds that to each pole in the σ-plane there corresponds a family of integrally spaced daughter poles in the complex J plane for the partial-wave amplitude a^J. This parent-daughter phenomenon anticipated in the works of Gribov and Volkov,[2] Domokos and Suranyi,[3] and rediscovered recently by Freedman and Wang,[4] finds its most complete expression in Toller's development insofar as, in contrast to the other authors, Toller takes full account of the very essential complications introduced by spin.

The aim of the present paper is to generalize Toller's work on elastic forward scattering and to expand the general two-body amplitude for all values of momentum transfer in terms of unitary representations of $SO(3,1)$. That such a program is feasible and that it may be expected to lead to new results has already been demonstrated by Oakes[5] and Domokos[6] for scattering of equal- or unequal-mass particles when no spins are involved. In this simple case, the amplitude is a function of scalar products of incoming and outgoing momenta. Such a function (or rather its analytic continuation to a Euclidean metric) can always be expanded in terms of a *complete set* of four-dimensional Gegenbauer polynomials.

Our concern in this paper is covariant inclusion of spin. One simple suggestion for doing this would be to separate out all spin-dependent factors and to write the general amplitude in terms of scalar amplitudes of

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[1] M. Toller, Nuovo Cimento 37, 631 (1965); Internal Reports University of Rome Nos. 76 (1965) and 84 (1965) (unpublished); M. Toller and A. Sciarrino, Internal Report, University of Rome No. 108 (1966) (unpublished).

[2] R. J. N. Volkov and V. N. Gribov, Zh. Eksperim. i Teor. Fiz. 44, 1068 (1963) [English transl.: Soviet Phys.—JETP 17, 720 (1963)].

[3] G. Domokos and P. Suranyi, Nucl. Phys. 54, 529 (1964); Ya. A. Smorodinski, M. Ulhir, and P. Winternitz, Dubna Report No. E-1591 (unpublished).

[4] D. Z. Freedman and J. M. Wang, Phys. Rev. 153, 1596 (1967); M. L. Goldberger and E. Jones, Phys. Rev. Letters 17, 105 (1966).

[5] R. Oakes, Phys. Letters 24B, 154 (1967).

[6] G. Domokos, Phys. Rev. 159, 1387 (1967).

the type used in Mandelstam's representation. For these one could certainly use the Oakes-Domokos-Gegenbauer expansion.

Unfortunately, this straightforward procedure suffers from two defects. Firstly, the choice of the amplitudes is arbitrary. Secondly, one would lose touch completely with Toller's development for forward scattering which employs the more physically transparent helicity formalism and which must serve as a necessary *boundary condition for any O(3,1) expansion of the general amplitude*. In this paper we find such an expansion, using conventional helicity formalism as far as possible.

Unfortunately, the formalism gets involved in a multitude of indices which somewhat obscure the simplicity of the basic ideas. Even at the risk of repetition we present a résumé of the argument of our paper in this introduction.

Consider scattering of two incoming particles of momenta, spins, and helicities p_i, S_i, and λ_i, $i=1\ 2$ to outgoing particles $i=3, 4$. Instead of the conventional S matrix, we choose to discuss the equivalent transition matrix T which takes $p_1,p_3 \to p_2,p_4$. This can be expressed in terms of three independent vectors $P=p_1-p_3=p_4-p_2$ (momentum transfer) and the relative momenta $q=p_1+p_3$, $q'=p_2+p_4$, in the form $(q';S_2\lambda_2,S_4\lambda_4|T(P)|q;S_1,\lambda_1,S_3,\lambda_3)$.

The first problem we face is that of covariant composition of S_1, λ_1 and S_3, λ_3. For Toller's case of zero momentum transfer, $P=0$ and $p_1=p_3$ so that one can simply "add" spins and helicities. In general, of course, the very definition of helicity ties momenta to spins and these must be decoupled before any "addition" is possible. Most of the prepossessing nature of our formulas comes about on account of this conceptually very simple step, with its standard procedure of passing from the helicity basis $|p,\lambda\rangle$ to the well-known[7] (dotted and undotted) spinor basis $|p,A\rangle$ and $|p,\dot{A}\rangle$. We thus employ, as it were, an M-function-like approach and pass from $|q;S_1\lambda_1,S_3\lambda_3\rangle$ to $|q;S_1A_1,S_3\dot{A}_3\rangle$ and then, finally, after a Clebsch-Gordan composition to "states" $|q,J,A\rangle$, reducing the problem to that of an expansion of $\langle q'J'\dot{A}'|T(P)|qJA\rangle$.

This is the step where Toller starts. For the forward-scattering situation ($P_\mu=0$), he notices that the state q' is created from state q by a simple Lorentz transformation. In the frame where we take $q+q'$ along t and $q-q'$ along the z axis, the q to q' transformation is an $O3$ rotation through the angle ζ between q and q'. Thus

$$\langle q'J'\dot{A}'|T(0)|qJA\rangle \sim \langle J'\dot{A}'|T(0)e^{-i\zeta J_{03}}|JA\rangle, \quad (1.1)$$

so that the expansion problem is simply the problem of writing matrix elements of the Lorentz rotation operator on the right of (1). These matrix elements are well known; they have been computed by Ström,[8] Toller,[1] Dao Wong Duc, and others, and are conventionally written as $D_{S'A',SA}{}^{j_0\sigma}(\zeta)=\delta_{A'A}d_{S'AS}{}^{j_0\sigma}(\zeta)$. The labels j_0 and σ correspond to the two invariants of $SO(3,1)$—one discrete (j_0) and the other continuous (σ). The T operator is a sum over j_0 and an integral over σ, and on account of the factor $\delta_{AA'}$ in the D function there is no helicity flip ($\lambda_1-\lambda_3=\lambda_2-\lambda_4$) as one should expect for forward scattering.

Now, although Toller has not considered this case, his considerations apply equally to the case when T is an invariant function of the four-vector P_μ [i.e., $T=\tilde{T}(P^2)$]. This is the key observation in what follows; linked with this is the observation that such matrix elements must be flipless.

A Tollerization of flipless amplitude is therefore always possible at all momentum transfers simply from the completeness relation satisfied by the functions $d_{S'AS}{}^{j_0\sigma}(\zeta)$. This expansion at $t\neq 0$ does not possess the same group-theoretic sanction as the $t=0$ case [$O(3,1)$ is not the invariance group any more]; however, since it passes automatically in the limit $t\to 0$ to the original Toller expansion (for pairwise equal masses) we believe that our expansion is equally as significant as Toller's.

For the spin-flip case we are not on such unambiguous ground. The procedure we have adopted is to express the spin-flip amplitudes as a series of kinematic terms[9] multiplied into (an equal number of) *flipless* amplitudes of the form $(\sqrt{(-P^2)})^{\text{flip}}(q'J'B|S(P^2)|qJB)$. These amplitudes are, as it were, half-way between helicity and scalar-invariant amplitudes and are constructed in Sec. 3. There can be more than one scheme of construction of such amplitudes; the scheme we have adopted has the merit that if the spin-flip amplitudes are approximated by a set of Toller trajectories, one can be sure that the parent Regge poles do lie on these.

The relationship of Regge formalism to that of Toller has been worked out for $t=0$ case by Toller and Sciarrino[1] and again by Freedman and Wang,[10] who continue the group $O(3,1)$ to $O(4)$ and use $O(4)$ rotation functions d^{kl} instead of $d^{j_0\sigma}$ for $O(3,1)$. The basic idea of these authors is to expand, for example, the $O(3,1)$ functions $d^{j_0\sigma}(\zeta)$ into a series of $O(2,1)$ functions $d^j(\zeta)$; this expansion gives rise to the parent-daughter phenomenon. For $t\neq 0$, we use the same idea and obtain not just parent-daughter poles at $t=0$ but parent-daughter trajectories for all t. An important consequence of the formalism is that the assumption of meromorphicity of amplitudes in σ plane—and this we must stress is a dynamical assumption—leads directly to the corresponding parent and daughter trajectories in the j plane being strictly parallel to each other.

One may therefore take the position that if daughter trajectories are experimentally discovered and found

[7] H. Joos, Fortschr. Physik **10**, 65 (1962).

[8] S. Ström, Arkiv Fysik **29**, 467 (1965); Dao Wong Duc and Nguyen Van Heiu, Dubna Report No. P-2886 (unpublished).

[9] A similar but not identical isolation of spin-flip factors from helicity amplitudes was carried out by M. Gell-Mann, M. L. Goldberger, F. E. Low, G. Marx, and F. Zachariasen [Phys. Rev. **133**, B145 (1964)] in defining their new amplitudes for Reggeization.

[10] D. Z. Freedman and J. M. Wang, Phys. Rev. **160**, 1560 (1967).

parallel or near parallel to their parents, the present formalism would provide a more desirable, a more compact, and (less singular) calculational procedure than the conventional one based on $O(3)$ expansions. If, on the other hand, no daughters are discovered, or if discovered, the trajectories are not nearly parallel, one must either approximate to the amplitudes by *families* of compensating Toller trajectories, or go back to the original Regge formalism with all its shortcomings at $t=0$. Since we believe that Toller's formalism possesses a firm group-theoretic foundation at $t=0$ and since ours (at least for the flipless case) is its direct continuation, it is unlikely that one will have to go back to the Regge formalism.[11]

The scheme of the present paper is as follows. In Sec. 2 we consider the problem of spin decoupling and pass from helicity amplitudes to M-function-like amplitudes. The material of the section is standard; the important end equation is (2.37) and the definition of the amplitude $\langle q'J'A'|T(P)|qJA\rangle$ through the relation (2.40). We state the basic expansion theorem (2.53), (2.54), which leads on the final expansion (2.58).

In Sec. 3 we consider the case of *flipping* amplitudes. Our aim is to define a set of independent flipless amplitudes in terms of which one can express these. Our procedure may be pictured graphically as the emission at *one* of the vertices (where particle p_1 comes in and p_3 goes out, for example) of a helicity-flip-carrying "spurion" which with helicity naturally also carries off spin, changing the spin $\mathbf{J}(=\mathbf{S}_1+\mathbf{S}_3)$ at this vertex to $J+1$, $J+2$, etc., $\cdots$ [Eqs. (3.17) and (3.18)]. Analytically the procedure is essentially one of the expansion of the T matrix $(q'|T(P)|q)$ in a power series $\sum P_\mu\cdots P_\nu(q'|T_{\mu\nu}\cdots(P_2)|q)$.

As stated earlier, there is an element of nonuniqueness about this expansion; one may expand, for example, in powers P_μ or the tensor $p_\mu q_\nu$ or a combination of these variables. In fact, in an earlier paper,[12] a different (and possibly simpler) expansion was suggested than the one adopted in Sec. 3. The difference between the two lies in the fact that the earlier version was tied to the center-of-mass (c.m.) frame. The formalism suggested here, on the contrary, is fully covariant, not tied to any given frame, and therefore also much more complicated. It must be stressed, however, that so far as Tollerization and the high-energy asymptotics are concerned, all expansions will give the same result since the behavior of $d_{j\mu j'}{}^{j_0\sigma}(z)$ for large z ($z^{\sigma-1-|j_0-\mu|}$) is independent of j and j', whereas it is just in the specification of j and j' that the alternative versions differ. The final result of the present paper is given in (3.17), (3.18), and (3.27).

One important departure in the present paper is the treatment of the unequal-mass case in Sec. 3, where, by a change of variables, this case is brought to correspond to the formalism for equal masses, thereby removing one of the vexatious points of recent work. The point we wish to stress is that it is of the essence of the factorizability theorem that the characteristics of a Toller or a Regge trajectory which is exchanged should not depend on whether the particles at a given vertex have equal or unequal masses.

We have not dealt here with the CTP properties of the amplitudes $(q'|T(P)|q)$. These can be inferred from the defining equation.

2. SPIN DECOUPLING

The representation of the Poincaré group associated with a physical particle of mass $m>0$, spin S, helicity λ, and positive energy is spanned by the set of basis vectors

$$\begin{aligned}|pS\lambda\rangle &= e^{-i\varphi J_{12}}e^{-i\theta J_{31}}e^{i\varphi J_{12}}e^{-i\alpha J_{03}}|\hat{p}S\lambda\rangle \\ &= U(L_p)|\hat{p}S\lambda\rangle\,,\end{aligned} \tag{2.1}$$

where

$$\hat{p}_\mu=(m,0,0,0)\,, \tag{2.2}$$

and

$$p_\mu=m(\cosh\alpha,\ \sinh\alpha\sin\theta\cos\varphi,\ \sinh\alpha\sin\theta\sin\varphi,\ \sinh\alpha\cos\theta)\,, \tag{2.3}$$

with

$$0\leqslant\varphi<2\pi\,,\quad 0\leqslant\theta\leqslant\pi\,,\quad 0\leqslant\alpha<\infty\,. \tag{2.4}$$

The $J_{\mu\nu}$ operators denote generators of homogeneous Lorentz transformations. The subset of states with $p=\hat{p}$ is defined by the $SO(3)$ conditions

$$\begin{aligned}\mathbf{J}^2|\hat{p}S\lambda\rangle &= S(S+1)|\hat{p}S\lambda\rangle\,,\\ J_{12}|\hat{p}S\lambda\rangle &= \lambda|\hat{p}S\lambda\rangle\,.\end{aligned} \tag{2.5}$$

Under homogeneous Lorentz transformations we have from Wigner's well-known theory,

$$U(\Lambda)|pS\lambda\rangle=\sum_{\lambda'}|\Lambda pS\lambda'\rangle D_{\lambda'\lambda}{}^S(L_{\Lambda p}{}^{-1}\Lambda L_p)\,. \tag{2.6}$$

The crucial point is that D^S is a representation of the $SO(3)$ little group; this form of (2.6) is possible because the transformation $L_{\Lambda p}{}^{-1}\Lambda L_p$ itself belongs to $SO(3)$.

One of the important ingredients of the development presented here turned out to be the necessity to decouple the p dependence of the spin transformations. The formalism for doing this is well known.[7] One modifies the basis system, the connection between the modified basis and the set (2.1) being provided by the $(2S+1)$-dimensional representations of the homogeneous group. In order to fix the notation, we shall review briefly some of the features of these representations.

It is usual in treating the finite-dimensional representations of $SO(3,1)$ to express the generators $J_{\mu\nu}$ in

[11] L. Durand [Phys. Rev. Letters 18, 58 (1967)] and R. Oakes (Ref. 5) have conjectured that there are no physical particles on daughter trajectories. In this view, Toller trajectories provide just the proper relativistic treatment for Regge trajectories without adding to the *physical-particle* spectrum.

[12] R. Delbourgo, Abdus Salam, and J. Strathdee, Phys. Letters **25B**, 230 (1967).

the form

$$J_{12}=K_3+L_3\,,\quad iJ_{03}=K_3-L_3\,,\quad \text{cyclically,} \tag{2.7}$$

where the operators $\mathbf{K}$ and $\mathbf{L}$ obey the commutation rules of $SO(3)\times SO(3)$, i.e.,

$$[K_1,K_2]=iK_3\,,\quad [L_1,L_2]=iL_3 \quad \text{cyclically}\,,$$

and

$$[K_i,L_j]=0\,. \tag{2.8}$$

The irreducible representations D^{kl} are then specified by a pair of integer or half-integer numbers (k,l), where

$$\mathbf{K}^2=k(k+1)\,,\quad \mathbf{L}^2=l(l+1)\,. \tag{2.9}$$

There are two alternative sets of basis vectors which are useful for expressing the finite-dimensional representations: one which diagonalizes K_3 and L_3 and one which diagonalizes $\mathbf{J}^2$ and J_{12}. These are written

$$|A,\dot{B}\rangle\,,$$
$$A=-k,\,-k+1,\,\cdots,\,k;\quad B=-l,\,-l+1,\,\cdots,\,l, \tag{2.10}$$

and

$$|j\lambda\rangle\,,$$
$$j=|k-l|,\,|k-l|+1,\,\cdots,\,k+l;\quad \lambda=-j,\,\cdots,j\,. \tag{2.11}$$

They satisfy the eigenvalue equations

$$K_3|A\dot{B}\rangle=A\,|A\dot{B}\rangle\,,\qquad L_3|A\dot{B}\rangle=B\,|A\dot{B}\rangle\,, \tag{2.12}$$

$$\mathbf{J}^2|j\lambda\rangle=j(j+1)\,|j\lambda\rangle\,,\quad J_{12}|j\lambda\rangle=\lambda\,|j\lambda\rangle\,. \tag{2.13}$$

Since $\mathbf{J}=\mathbf{K}+\mathbf{L}$, the connection between these basis systems is provided by the ordinary Clebsch-Gordan coefficients.

$$\begin{aligned}|j\lambda\rangle&=\sum_{AB}|A\dot{B}\rangle\langle kA,lB\,|\,j\lambda\rangle\,,\\ |A\dot{B}\rangle&=\sum_{j\lambda}|j\lambda\rangle\langle j\lambda\,|\,kA,lB\rangle\,.\end{aligned} \tag{2.14}$$

We give examples of finite transformations in the $|A\dot{B}\rangle$ system. Firstly, corresponding to a purely spatial rotation,

$$\begin{aligned}U(\varphi,\theta,\psi)&=e^{-i\varphi J_{12}}e^{-i\theta J_{31}}e^{-i\psi J_{12}}\\ &=e^{-i\varphi K_3}e^{-i\theta K_2}e^{-i\psi K_3}e^{-i\varphi L_3}e^{-i\theta L_2}e^{-i\psi L_3}\,,\end{aligned} \tag{2.15}$$

we have

$$U(\varphi,\theta,\psi)\,|A\dot{B}\rangle=\sum_{A'B'}|A'\dot{B}'\rangle \times D_{A'A}{}^{k}(\varphi,\theta,\psi)D_{B'B}{}^{l}(\varphi,\theta,\psi)\,, \tag{2.16}$$

where $D_{A'A}{}^k$ and $D_{B'B}{}^l$ denote the usual $SO(3)$ rotation matrices

$$D_{A'A}{}^k(\varphi,\theta,\psi)=e^{-iA'\varphi}d_{A'A}{}^k(\theta)e^{-iA\psi}\,, \tag{2.17}$$

and similarly for $D_{B'B}{}^l$. It will suffice to exhibit only the pure Lorentz transformation in the 03 plane since any other can be built up from this one together with appropriate spatial rotations. Using $iJ_{03}=K_3-L_3$, we get

$$e^{-i\alpha J_{03}}|A\dot{B}\rangle=|A\dot{B}\rangle e^{-\alpha(A-B)}\,. \tag{2.18}$$

It is a simple matter to express these transformations in the $|j\lambda\rangle$ basis. Thus,

$$U(\varphi,\theta,\psi)\,|j\lambda\rangle=\sum_{\lambda'}|j\lambda'\rangle D_{\lambda'\lambda}{}^{j}(\varphi,\theta,\psi)\,, \tag{2.19}$$

$$e^{-i\alpha J_{03}}|j\lambda\rangle=\sum_{j'}|j'\lambda\rangle d_{j'\lambda j}{}^{kl}(\alpha)\,, \tag{2.20}$$

where

$$d_{j'\lambda j}{}^{kl}(\alpha)=\sum_{AB}\langle j'\lambda\,|\,kA,lB\rangle e^{-\alpha(A-B)}\langle kA,lB\,|\,j\lambda\rangle\,. \tag{2.21}$$

Of particular interest to us in the following are the representations D^{S0} and D^{0S} which are *irreducible under* $SO(3)$. Since the Clebsch-Gordan coefficients reduce to Kronecker symbols in these cases, the distinction between the bases (2.10) and (2.11) disappears. For the finite transformations dealt with above we have

$$\begin{aligned}D^{S0}\colon\;& U(\varphi,\theta,\psi)\,|SA\rangle=\sum_{A'}|SA'\rangle D_{A'A}{}^{S}(\varphi,\theta,\psi)\,,\\ & e^{-i\alpha J_{03}}|SA\rangle=|SA\rangle e^{-\alpha A};\end{aligned} \tag{2.22}$$

$$\begin{aligned}D^{0S}\colon\;& U(\varphi,\theta,\psi)\,|S\dot{A}\rangle=\sum_{A'}|S\dot{A}'\rangle D_{A'A}{}^{S}(\varphi,\theta,\psi)\,,\\ & e^{-i\alpha J_{03}}|S\dot{A}\rangle=|S\dot{A}\rangle e^{\alpha A}\,.\end{aligned} \tag{2.23}$$

Finally, let us remark that the decomposition of a direct product of two irreducible representations is little more complicated than the corresponding problem in $SO(3)$. Thus,

$$D^{k_1l_1}\otimes D^{k_2l_2}=\sum_{k=|k_1-k_2|}^{k_1+k_2}\;\sum_{l=|l_1-l_2|}^{l_1+l_2}D^{kl}\,, \tag{2.24}$$

or, in terms of basis vectors,

$$\begin{aligned}&|k_1A_1l_1\dot{B}_1,k_2A_2l_2\dot{B}_2\rangle\\ &\quad=\sum_{kl}|k_1k_2kA,l_1l_2l\dot{B}\rangle\\ &\qquad\times\langle kA\,|\,k_1A_1,k_2A_2\rangle\langle lB\,|\,l_1B_1,l_2B_2\rangle\,,\end{aligned}$$

or

$$\begin{aligned}&|k_1l_1j_1\lambda_1,k_2l_2j_2\lambda_2\rangle\\ &\quad=\sum_{klj}|k_1l_1k_2l_2,klj\lambda\rangle\langle j\lambda\,|\,j_1\lambda_1j_2\lambda_2\rangle\\ &\qquad\times\langle (k_1l_1)j_1,(k_2l_2)j_2;\,j\,|\,(k_1k_2)k,(l_1l_2)l;\,j\rangle\,,\end{aligned} \tag{2.25}$$

where, in the last expression, a $9-j$ symbol has made its appearance.

Returning now to the problem of decoupling the p dependence in the transformation law (2.6) we see that since D^{S0} and D^{0S} are irreducible under spatial rotations, it is always possible to write

$$D_{\lambda'\lambda}{}^{S}(L_{\Lambda p}{}^{-1}\Lambda L_p)=D_{\lambda'\lambda}{}^{S0}(L_{\Lambda p}{}^{-1}\Lambda L_p)=\sum_{AA'}D_{\lambda'A'}{}^{S0}(L_{\Lambda p}{}^{-1})D_{A'A}{}^{S0}(\Lambda)D_{A\lambda}{}^{S0}(L_p)\,, \quad (2.26)$$

or, alternatively,

$$D_{\lambda'\lambda}{}^{S}(L_{\Lambda p}{}^{-1}\Lambda L_p)=D_{\lambda'\lambda}{}^{0S}(L_{\Lambda p}{}^{-1}\Lambda L_p)=\sum_{AA'}D_{\lambda'A'}{}^{0S}(L_{\Lambda p}{}^{-1})D_{A'A}{}^{0S}(\Lambda)D_{A\lambda}{}^{0S}(L_p)\,. \quad (2.27)$$

Let us therefore define the modified basis systems

$$|pSA\rangle=\sum_{\lambda}|pS\lambda\rangle D_{\lambda A}{}^{S0}(L_p{}^{-1})\,, \quad (2.28)$$

and

$$|pS\dot{A}\rangle=\sum_{\lambda}|pS\lambda\rangle D_{\lambda A}{}^{0S}(L_p{}^{-1})\,. \quad (2.29)$$

It follows immediately from (2.6) that under a homogeneous Lorentz transformation these states transform, respectively, according to the laws

$$U(\Lambda)|pSA\rangle=\sum_{A'}|\Lambda pSA'\rangle D_{A'A}{}^{S0}(\Lambda)\,, \quad (2.30)$$

$$U(\Lambda)|pS\dot{A}\rangle=\sum_{A'}|\Lambda pS\dot{A}'\rangle D_{A'A}{}^{0S}(\Lambda)\,. \quad (2.31)$$

As it happens, the set (2.29) transforms contragrediently to the set (2.28). This follows from the nonunitary nature of the finite-deminsional representations

$$[D_{AB}{}^{S0}(\Lambda)]^*=D_{BA}{}^{0S}(\Lambda^{-1})\,. \quad (2.32)$$

If we adopt the notational convention

$$\langle\,|pSA\rangle^*=\langle pS\dot{A}|\,\rangle\,,\quad \langle\,|pS\dot{A}\rangle^*=\langle pSA|\,\rangle\,,$$

then, corresponding to (2.28) and (2.29), respectively, we must have

$$\langle pS\dot{A}|=\sum_{\lambda}D_{A\lambda}{}^{0S}(L_p)\langle pS\lambda|\,, \quad (2.33)$$

$$\langle pSA|=\sum_{\lambda}D_{A\lambda}{}^{S0}(L_p)\langle pS\lambda|\,. \quad (2.34)$$

Thus, in particular, the orthogonality condition reads

$$\delta(p^2-m^2)\langle pSA|p'SB\rangle=\delta_{AB}\delta(p-p')\,.$$

Let us note finally the explicit form taken by the matrices in (2.28) and (2.29) when L_p refers to the boost defined in (2.1);

$$|pSA\rangle=\sum_{\lambda}|pS\lambda\rangle e^{-i\lambda(\varphi+i\alpha)}d_{\lambda A}{}^{S}(-\theta)e^{iA\varphi}, \quad (2.35)$$

$$|pS\dot{A}\rangle=\sum_{\lambda}|pS\lambda\rangle e^{-i\lambda(\varphi-i\alpha)}d_{\lambda A}{}^{S}(-\theta)e^{iA\varphi}. \quad (2.36)$$

The application for which the above formalism has been developed concerns the analysis of T-matrix elements. Let us consider the process

$$\langle p_3S_3\lambda_3,p_4S_4\lambda_4|T|p_1S_1\lambda_1,p_2S_2\lambda_2\rangle\,,$$

where the λ's denote helicity. From this matrix element we can define a new one by

$$\langle p_3S_3\lambda_3,p_4S_4\lambda_4|T|p_1S_1\lambda_1,p_2S_2\lambda_2\rangle=\sum_{A_1\cdots A_4}D_{\lambda_3A_3}{}^{S_30}(L_{p_3}{}^{-1})D_{\lambda_4A_4}{}^{0S_4}(L_{p_4}{}^{-1})\times\langle p_3S_3A_3,p_4S_4\dot{A}_4|T|p_1S_1A_1,p_2S_2\dot{A}_2\rangle\times D_{A_1\lambda_1}{}^{S_10}(L_{p_1})D_{A_2\lambda_2}{}^{0S_2}(L_{p_2})\,. \quad (2.37)$$

The newly defined matrix element, which is a type of M function, satisfies the invariance condition

$$\langle p_3S_3A_3,p_4S_4\dot{A}_4|T|p_1S_1A_1p_2S_2\dot{A}_2\rangle=\sum_{B_1\cdots B_4}D_{A_3B_3}{}^{S_30}(\Lambda^{-1})D_{A_4B_4}{}^{0S_4}(\Lambda^{-1})\times\langle\Lambda p_3S_3B_3,\Lambda p_4S_4\dot{B}_4|T|\Lambda p_1S_1B_1,\Lambda p_2S_2\dot{B}_2\rangle\times D_{B_1A_1}{}^{S_10}(\Lambda)D_{B_2A_2}{}^{0S_2}(\Lambda)\,. \quad (2.38)$$

A more compact and convenient notation for this matrix element can be introduced if we define the three independent vectors

$$\begin{aligned}P&=p_1-p_3=p_4-p_2\,,\\2q&=p_1+p_3\,,\\2q&=p_2+p_4\,.\end{aligned} \quad (2.39)$$

Let us therefore couple the spins S_1S_3 and S_2S_4, writing

$$\langle p_3S_3A_3,p_4S_4\dot{A}_4|T|p_1S_1A_1p_2S_2\dot{A}_2\rangle=\sum_{JJ'}\langle S_4A_4|J'A',S_2A_2\rangle\langle q'J'\dot{A}'|T(P)|qJA\rangle\times\langle S_3A_3,JA|S_1A_1\rangle\,. \quad (2.40)$$

The matrix element $\langle q'J'\dot{A}'|T(P)|qJA\rangle$ satisfies the invariance condition

$$\langle q'J'\dot{A}'|T(P)|qJA\rangle=\sum_{BB'}D_{A'B'}{}^{0J'}(\Lambda^{-1})\times\langle\Lambda qJ'\dot{B}'|T(\Lambda P)|\Lambda qJB\rangle D_{BA}{}^{J0}(\Lambda) \quad (2.41)$$

for any homogeneous Lorentz transformation Λ. An equivalent formulation of this requirement is contained in the three relations

$$U(\Lambda)|qJA\rangle=\sum_{B}|\Lambda qJB\rangle D_{BA}{}^{J0}(\Lambda)\,,$$

$$\langle q'J'\dot{A}'|U^{-1}(\Lambda)=\sum_{B'}D_{A'B'}{}^{0J}(\Lambda^{-1})\langle\Lambda q'J'\dot{B}'|\,, \quad (2.42)$$

$$U(\Lambda)T(P)U^{-1}(\Lambda)=T(\Lambda P)\,.$$

Let us denote by $L_q{}^{-1}$ and $L_{q'}{}^{-1}$ two transformations which carry q and q', respectively, into the time axis

$$L_q^{-1}q=\hat{q} \quad\text{and}\quad L_{q'}^{-1}q'=\hat{q}', \tag{2.43}$$

where $\hat{q}$ and $\hat{q}'$ have vanishing space components. With these notations we can write formally

$$\begin{aligned}\langle q'J'A'|T(P)|qJA\rangle &=\sum_{BB'} D_{A'B'}{}^{0J'}(L_{q'})\langle\hat{q}'J'\dot{B}'|U(L_{q'}^{-1})T(P)\\ &\qquad\times U(L_q)|\hat{q}JB\rangle D_{BA}{}^{J0}(L_q^{-1})\\ &=\sum_{BB'} D_{A'B'}{}^{0J'}(L_{q'})\langle\hat{q}'J'\dot{B}'|T(L_{q'}^{-1}P)\\ &\qquad\times U(L_{q'}^{-1}L_q)|\hat{q}JB\rangle D_{BA}{}^{J0}(L_q^{-1}),\end{aligned} \tag{2.44}$$

thereby expressing the matrix element as a function of two Lorentz transformations.

Now our principal aim in this paper is to develop the matrix element in irreducible unitary representations of the homogeneous Lorentz group. This can be done by standard group-theoretical techniques provided that we reduce it to a function of only one Lorentz transformation. In Sec. III a method is presented for separating out the angular dependence on P_μ, and once this has been done we shall have, in effect,

$$T(\Lambda P)=T(P).$$

For the special circumstance of forward scattering ($P_\mu=0$) this is of course unnecessary—the expression (2.39) already contains only one Lorentz transformation.

Let us assume for the present that $T=T(P^2)$ and so proceed with the development. For the matrices of irreducible unitary representations of the Lorentz group we adopt the notation—in a $|j\lambda\rangle$ basis—

$$\begin{aligned}D_{j\lambda,j'\lambda'}{}^{j_0\sigma}(\Lambda):\quad & j,j'=|j_0|,\,|j_0|+1,\,|j_0|+2,\,\cdots,\\ &\lambda=-j,\cdots,j,\\ &\lambda'=-j',\cdots,j'.\end{aligned} \tag{2.45}$$

The labels j_0 and σ are given in terms of the two Casimir invariants by

$$\begin{aligned}\tfrac{1}{2}J_{\mu\nu}J_{\mu\nu}&=j_0^2+\sigma^2-1,\\ \tfrac{1}{4}\epsilon_{\lambda\mu\nu\rho}J_{\lambda\mu}J_{\nu\rho}&=2ij_0\sigma.\end{aligned} \tag{2.46}$$

The representations of the principal series, which are the only ones we shall need, are given by j_0 and σ in the ranges

$$\begin{aligned}\pm j_0&=0,\tfrac{1}{2},1,\tfrac{3}{2},\cdots,\\ 0&\leqslant -i\sigma<\infty.\end{aligned} \tag{2.47}$$

The structure of the matrices (2.40) is such that for purely spatial rotations R we have

$$D_{j\lambda,j'\lambda'}{}^{j_0\sigma}(R)=\delta_{jj'}D_{\lambda\lambda'}{}^{j}(R), \tag{2.48}$$

where $D_{\lambda\lambda'}{}^{j}$ denotes one of the usual $SO(3)$ rotation matrices (2.17). Corresponding to the pure Lorentz transformation in the 03 plane through a hyperbolic angle [see (2.20)], we write[13]

$$D_{j\lambda,j'\lambda'}{}^{j_0\sigma}(\alpha)=\delta_{\lambda\lambda'}d_{j\lambda j'}{}^{j_0\sigma}(\alpha). \tag{2.49}$$

The group-theoretical result which we shall exploit is the theorem which states that any square-integrable function over the Lorentz group may be expanded in the representations of the principal series. Let $f(\Lambda)$ be such a function. If we define the coefficients

$$f_{j\lambda,j'\lambda'}{}^{j_0\sigma}=\int d\Lambda\, D_{j'\lambda',j\lambda}{}^{j_0\sigma}(\Lambda)^*f(\Lambda), \tag{2.50}$$

where $\int d\Lambda$ denotes the invariant integral over the group, then we can make the expansion

$$f(\Lambda)=\sum_{j_0}\int_0^{i\infty}d\sigma(j_0^2-\sigma^2)\times\sum_{j\lambda,j'\lambda'}f_{j\lambda,j'\lambda'}{}^{j_0\sigma}D_{j\lambda,j'\lambda'}{}^{j_0\sigma}(\Lambda). \tag{2.51}$$

Applying this to the function $\langle\hat{q}S'\dot{B}'|T(P^2)U(\Lambda)|\hat{q}SB\rangle$ and using in addition the properties

$$\begin{aligned}U(R)|\hat{q}JB\rangle&=\sum_A|\hat{q}JA\rangle D_{AB}{}^J(R),\\ \langle\hat{q}J'\dot{B}'|U^{-1}(R)&=\sum_{A'}D_{B'A'}{}^J(R^{-1})\langle\hat{q}J'\dot{A}'|,\end{aligned} \tag{2.52}$$

for purely spatial rotations, we get

$$\int d\Lambda\, D_{j'\lambda',j\lambda}{}^{j_0\sigma}(\Lambda)^*\langle\hat{q}J'\dot{B}'|T(P^2)U(\Lambda)|\hat{q}JB\rangle =\delta_{j'J'}\delta_{\lambda'B'}\delta_{jJ}\delta_{\lambda B}\langle J'|T^{j_0\sigma}(P^2)|J\rangle, \tag{2.53}$$

and, conversely,

$$\langle\hat{q}J'\dot{B}'|T(P^2)U(\Lambda)|\hat{q}JB\rangle =\sum_{j_0=-M}^{M}\int_0^{i\infty}d\sigma(j_0^2-\sigma^2)\langle J'|T^{j_0\sigma}(P^2)|J\rangle\times D_{J'B'JB}{}^{j_0\sigma}(\Lambda), \tag{2.54}$$

where M denotes the lesser of J and J'. Inserting the expansion (2.49) into (2.40) we get

$$\begin{aligned}\langle q'J'A'|T(P^2)|qJA\rangle &=\sum_{j_0=-M}^{M}\int_0^{i\infty}d\sigma(j_0^2-\sigma^2)\langle J'|T^{j_0\sigma}(P^2)|J\rangle\\ &\quad\times\sum_{BB'}D_{A'B'}{}^{0J'}(L_{q'})D_{J'B',JB}{}^{j_0\sigma}(L_{q'}^{-1}L_q)\\ &\quad\times D_{BA}{}^{J0}(L_q^{-1}).\end{aligned} \tag{2.55}$$

In a suitably chosen frame it is possible to fix L_q and

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[13] In the papers of Toller, $(j_0,\sigma)\to(M,\lambda)$. Our notation in (2.49) differs slightly in that Toller writes

$$D_{j\mu,j'\mu'}{}^{M\lambda}(\alpha)=\delta_{\mu\mu'}d_{\mu jj'}{}^{M\lambda}(\alpha).$$

$L_{q'}$ by

$$U(L_q)=e^{-\frac{1}{2}i\zeta J_{03}},\quad U(L_{q'})=e^{\frac{1}{2}i\zeta J_{03}}e^{-i\pi J_{31}},\quad (2.56)$$

in which case we have

$$\sum_{BB'} D_{A'B'}{}^{0J'}(L_{q'})D_{J'B',JB}{}^{j_0\sigma}(L_{q'}{}^{-1}L_q)D_{BA}{}^{J0}(L_q{}^{-1})$$
$$=\delta_{A'A}e^{-\zeta A'/2}d_{J'AJ}{}^{j_0\sigma}(\zeta)e^{\zeta A/2}.\quad (2.57)$$

In this frame the formula (2.51) becomes

$$\langle q'J'\dot{A}'|T(P^2)|qJA\rangle$$
$$=\delta_{A'A}\sum_{j_0=-M}^{M}\int_0^{i\infty}d\sigma(j_0{}^2-\sigma^2)\langle J'|T^{j_0\sigma}(P^2)|J\rangle \times d_{J'AJ}{}^{j_0\sigma}(\zeta),\quad (2.58)$$

where ζ denotes the angle between q and q'

$$\cosh\zeta=qq'/(q^2q'^2)^{1/2}.\quad (2.59)$$

3. REDUCTION OF THE MATRIX ELEMENTS; THE SPIN-FLIP CASE

In order to carry out the expansion in irreducible representations of the matrix element $\langle q'J'\dot{A}'|T(P)|qJA\rangle$, it is necessary first to separate out the dependence on the angular coordinates of P_μ. This separation corresponds in part to an expansion of the matrix element in invariant amplitudes. However, we shall not carry out a complete reduction to invariant amplitudes but rather we shall separate only the powers of P_μ. The coefficients in this expansion we shall denote by $\langle q'j'\dot{B}'|T(P^2)|qjB\rangle$. It is these simpler objects which depend on P^2 rather than P_μ which can be developed in irreducible representations of the homogeneous Lorentz group. We may expect, of course, that our method of isolating the P_μ dependence has the feature—common to all expansions in invariant amplitudes—of being nonunique.

The problem of separating out the powers of P_μ is made a little simpler if we deal instead with the self-dual antisymmetric tensor.

$$f_{\mu\nu}=\tfrac{1}{3}\sqrt{3}[(g_{\mu\lambda}g_{\nu\rho}-g_{\nu\lambda}g_{\mu\rho})+i\epsilon_{\mu\nu\lambda\rho}]P_\lambda(q+q')_\rho,\quad (3.1)$$

which is linear in P_μ. Thus, we propose to write the matrix elements of $T(P)$ in the form

$$\langle q'J'\dot{A}'|T(P)|qJA\rangle$$
$$=\sum_{jB}\langle q'J'\dot{A}'|T(P^2)|qJjB\rangle\langle jB|\Gamma(f)|JA\rangle,\quad (3.2)$$

where $\Gamma(f)$ denoted a suitable polynomial in $f_{\mu\nu}$. The summation variables j, B must cover such a range as to give the correct count of reduced amplitudes. Disregarding the constraints arising from the discrete T, C, and P symmetries, there are just $(2J+1)(2J'+1)$ independent matrix elements of $T(P)$—one for each value of A, A'. On the other hand, for fixed j there are only $\min[(2J'+1),(2j+1)]$ independent matrix elements of $T(P^2)$. One way of getting the correct total is to let j take the values

$$J',\ J'+1,\ \cdots,\ J'+2J.\quad (3.3)$$

Since $j\geqslant J'$, and since there are $2J+1$ values, we have

$$\sum_j \min[(2J'+1)(2j+1)]=\sum_j(2J'+1)$$
$$=(2J+1)(2J'+1).$$

It remains to choose the matrix elements of $\Gamma(f)$ in the simplest possible way consistent with the covariance of (3.2). A further requirement to be imposed is that the formula (3.2) reduce at $P_\mu=0$ to

$$\langle q'J'\dot{A}'|T(0)|qJA\rangle=\langle q'J'\dot{A}'|T(0)|qJJA\rangle.\quad (3.4)$$

From the covariance conditions

$$\langle q'J'A'|T(P)|qJA\rangle$$
$$=\sum_{BB'}D_{A'B'}{}^{0J'}(\Lambda^{-1}) \times\langle\Lambda q'J'\dot{B}'|T(\Lambda P)|\Lambda qJB\rangle D_{BA}{}^{J0}(\Lambda),\quad (3.5)$$

$$\langle q'J'\dot{A}'|T(P^2)|qJjA\rangle$$
$$=\sum_{BB'}D_{A'B'}{}^{0J'}(\Lambda^{-1}) \times\langle\Lambda q'J'\dot{B}'|T(P^2)|\Lambda qJjB\rangle D_{BA}{}^{j0}(\Lambda),\quad (3.6)$$

it follows that the covariance of (3.2) requires $\Gamma(f)$ to satisfy the covariance condition

$$\langle jB|\Gamma(f)|JA\rangle=\sum_{CD}D_{BC}{}^{j0}(\Lambda^{-1}) \times\langle jC|\Gamma(\Lambda f)|JD\rangle D_{DA}{}^{J0}(\Lambda),\quad (3.7)$$

where, of course,

$$(\Lambda f)_{\mu\nu}=\Lambda_{\mu\rho}\Lambda_{\nu\sigma}f_{\rho\sigma}.\quad (3.8)$$

We can use (3.7) to simplify the structure of $\Gamma(f)$.

The self-dual tensor $f_{\mu\nu}$ has three independent components. Let us write

$$(f_1,f_2,f_3)=(f_{01},f_{02},f_{03})=(if_{23},if_{31},if_{12}).\quad (3.9)$$

For an arbitrary Λ we have

$$(\Lambda f)_j=\sum_k D_{jk}{}^{10}(\Lambda)f_k,\quad (3.10)$$

while

$$f^2=\sum_j f_jf_j=-\tfrac{1}{4}f_{\mu\nu}f_{\mu\nu}\quad (3.11)$$

is invariant. We are assuming that $\Gamma(f)$ is a polynomial in f_j,

$$\Gamma(f)=\sum_n\Gamma_{j_1\cdots j_n}f_{j_1}\cdots f_{j_n}.\quad (3.12)$$

The coefficients $\Gamma_{j_1\cdots j_n}$ transform as tensors under homogeneous Lorentz transformations and they can be decomposed into irreducible parts. Choosing Λ in (3.7) such that $\Lambda f=\hat{f}=(0,0,\sqrt{(f^2)})$ we can write

$$\begin{aligned}\langle jB|\Gamma(f)|JA\rangle &=\sum_{CD} D_{BC}{}^{j0}(\Lambda^{-1})\langle jC|\Gamma(\hat{f})|JD\rangle D_{DA}{}^{J0}(\Lambda)\\ &=\sum_{CD}\sum_{n} D_{BC}{}^{j0}(\Lambda^{-1})\langle jC|f^n\Gamma_0{}^n(f^2)|JD\rangle D_{DA}{}^{J0}(\Lambda)\\ &=\sum_{CD}\sum_{n} D_{BC}{}^{j0}(\Lambda^{-1})f^n\langle j\|\Gamma^n\|J\rangle\langle jC|n0,JD\rangle D_{DA}{}^{J0}(\Lambda)\\ &=\sum_{nC} f^n\langle j\|\Gamma^n\|J\rangle\langle jB|nC,JA\rangle D_{C0}{}^{n0}(\Lambda^{-1})\\ &=\sum_{nC} f^n\langle j\|\Gamma^n\|J\rangle\langle jB|nC,JA\rangle e^{-iC\beta}d_{C0}{}^n(\alpha)\,, \end{aligned}\tag{3.13}$$

where α and β denote the polar coordinates of f_j;

$$f_j=(f^2)^{1/2}(\sin\alpha\,\cos\beta,\,\sin\alpha\,\sin\beta,\,\cos\alpha)\,.\tag{3.14}$$

The reduced matrix elements $\langle j\|\Gamma^n\|J\rangle$ must be chosen in some convenient way. Let us require that only the smallest possible value of n survives in the sum (3.13), i.e., let us take

$$\langle j\|\Gamma^n\|J\rangle=\delta_{n,|J-j|}\,.\tag{3.15}$$

The matrix $\Gamma(f)$ is now completely specified, viz.,

$$\langle jB|\Gamma(f)|JA\rangle =f^{|J-j|}\langle jB|\,|J-j|\,\Delta,JA\rangle e^{-i\Delta\beta}d_{\Delta 0}{}^{|J-j|}(\alpha)\,,\tag{3.16}$$

where $\Delta=B-A$. The expression (3.2) thus takes the form

$$\begin{aligned}\langle q'J'\dot{A}'|T(P)|qJA\rangle &=\sum_{j=J'}^{J'+2J}\sum_{B} f^{|J-j|}\langle q'J'\dot{A}'|T(P^2)|qJjB\rangle\\ &\qquad\times\langle jB|\,|J-j|\,\Delta,JA\rangle e^{-i\Delta\beta}d_{\Delta 0}{}^{|J-j|}(\alpha)\,.\end{aligned}\tag{3.17}$$

This form satisfies the requirement (3.4) only for $J'\leqslant J$. Matrix elements with $J'>J$ must be dealt with by extracting powers of f^*. Arguments similar to those used above lead to the form

$$\begin{aligned}\langle q'J'\dot{A}'|T(P)|qJA\rangle &=\sum_{j'B'}\langle J'A'|\Gamma(f^*)|j'\dot{B}'\rangle\langle q'J'j'\dot{B}'|T(P^2)|qJA\rangle\\ &=\sum_{j'=J}^{J+2J'}\sum_{B'} f^{|J'-j'|}\langle J'A'|\,|J'-j'|\,\Delta',j'B'\rangle\\ &\quad\times e^{-i\beta^*\Delta'}d_{\Delta'0}{}^{|J'-j'|}(\alpha^*)\langle q'J'j'\dot{B}'|T(P^2)|qJA\rangle\,,\end{aligned}\tag{3.18}$$

where $\Delta'=A'-B'$.

The most convenient frame, for practical purposes, is the one specified by (2.56). In this frame the components of $f_{\mu\nu}$ can be expressed as functions of the Mandelstam invariants s and t. From (3.1) and (3.9),

$$\begin{aligned}s^{1/2}f_1&=-P_1(q+q')_0\,,\\ s^{1/2}f_2&=-iP_1(q+q')_3\,,\\ s^{1/2}f_3&=P_0(q+q')_3-P_3(q+q')_0\,;\end{aligned}\tag{3.19}$$

and from (2.39) and (2.56),

$$2q_\mu=[2(m_1{}^2+m_3{}^2)-t]^{1/2}(\cosh\tfrac{1}{2}\zeta,\,0,\,0,\,\sinh\tfrac{1}{2}\zeta)\,,\tag{3.20}$$

$$2q_\mu'=[2(m_2{}^2+m_4{}^2)-t]^{1/2}(\cosh\tfrac{1}{2}\zeta,\,0,\,0,\,-\sinh\tfrac{1}{2}\zeta)\,,$$

$$P_\mu=(P_0,P_1,0,P_3)\,,$$

where

$$P_0=\frac{1}{2\cosh\frac{1}{2}\zeta}\left[\frac{m_4{}^2-m_2{}^2}{(2(m_4{}^2+m_2{}^2)-t)^{\frac{1}{2}}}+\frac{m_1{}^2-m_3{}^2}{(2(m_1{}^2+m_3{}^2)-t)^{\frac{1}{2}}}\right],$$

$$P_3=\frac{1}{2\sinh\frac{1}{2}\zeta}\left[\frac{m_4{}^2-m_2{}^2}{(2(m_4{}^2+m_2{}^2)-t)^{\frac{1}{2}}}-\frac{m_1{}^2-m_3{}^2}{(2(m_1{}^2+m_3{}^2)-t)^{\frac{1}{2}}}\right],$$

$$P_1=[P_0{}^2-P_3{}^2-t]^{1/2}\,,$$

and

$$s-u=[(2(m_1{}^2+m_3{}^2)-t)\times(2(m_4{}^2+m_2{}^2)-t)]^{1/2}\cosh\zeta\,.\tag{3.21}$$

The invariant f is given by

$$\begin{aligned}sf^2&=P_1{}^2(q+q')_0{}^2-P_1{}^2(q+q')_3{}^2+(P_0(q+q')_3-P_3(q+q')_0)^2\\ &=[P\cdot(q+q')]^2-P^2(q+q')^2\,,\\ f^2&=(1/4s)[m_1{}^2-m_2{}^2-m_3{}^2+m_4{}^2]^2-t\,.\end{aligned}\tag{3.22}$$

For the elastic case $m_1=m_3=M$, $m_2=m_4=\mu$, we have

$$\begin{aligned}s^{1/2}f_1&=-\tfrac{1}{2}(-t)^{1/2}[(4M^2-t)^{1/2}+(4\mu^2-t)^{1/2}]\cosh\tfrac{1}{2}\zeta\,,\\ s^{1/2}f_1&=-\tfrac{1}{2}i(-t)^{1/2}[(4M-t)^{1/2}-(4\mu^2-t)^{1/2}]\sinh\tfrac{1}{2}\zeta\,,\\ s^{1/2}f_3&=0\,,\end{aligned}\tag{3.23}$$

so that

$$\alpha=\tfrac{1}{2}\pi\,,$$

$$\tan\beta=i\left(\frac{(4M^2-t)^{\frac{1}{2}}-(4\mu^2-t)^{\frac{1}{2}}}{(4M^2-t)^{\frac{1}{2}}+(4\mu^2-t)^{\frac{1}{2}}}\right)\tanh\tfrac{1}{2}\zeta\,.\tag{3.24}$$

An alternative separation of the powers of P_μ can be made if we make the replacement in (3.1),

$$P_\mu\to P_\mu'=P_\mu-\gamma q_\mu-\gamma' q_\mu'\,,$$

where γ and γ' are chosen so as to bring P_μ' into the form

$$P_\mu'=(0,P_1',0,0)\,.\tag{3.25}$$

They are

$$\gamma=\frac{(m_1{}^2-m_3{}^2)q'^2-(m_4{}^2-m_2{}^2)qq'}{q^2q'^2-(qq')^2}\,,$$

$$\gamma'=\frac{(m_4{}^2-m_2{}^2)q^2-(m_1{}^2-m_3{}^2)qq'}{q^2q'^2-(qq')^2}\,.$$

With this choice we have

$$\alpha=\tfrac{1}{2}\pi\,,$$

$$\tan\beta=\left[\frac{(2(m_1{}^2+m_3{}^2)-t)^{\frac{1}{2}}-(2(m_4{}^2+m_2{}^2)-t)^{\frac{1}{2}}}{(2(m_1{}^2+m_3{}^2)-t)^{\frac{1}{2}}+(2(m_4{}^2+m_2{}^2)-t)^{\frac{1}{2}}}\right]\tan\tfrac{1}{2}i\zeta,$$

and

$$f^2=-P'^2. \tag{3.26}$$

The development in irreducible representations of the helicity amplitudes is given by

$$\begin{aligned}\langle p_3\lambda_3,p_4\lambda_4|T|p_1\lambda_1,p_2\lambda_2\rangle &= \sum_{A_1\cdots A_4}\sum_{JA}\sum_{J'A'} D_{\lambda_4A_4}{}^{0S_4}(L_{p_4}{}^{-1})\\ &\quad\times\langle S_4A_4|J'A',S_2A_2\rangle D_{A_2\lambda_2}{}^{0S_2}(L_{p_2})\\ &\quad\times\langle q'J'A'|T(P)|qJA\rangle D_{\lambda_3A_3}{}^{S_30}(L_{p_3}{}^{-1})\\ &\quad\times\langle S_3A_3,JA|S_1A_1\rangle D_{A_1\lambda_1}{}^{S_10}(L_{p_1})\,,\end{aligned} \tag{3.27}$$

where $\langle q'J'A'|T(P)|qJA\rangle$ is expanded according to (3.17) for $J'\leqslant J$ or, according to (3.18) for $J'\geqslant J$.

The boost matrices here are defined by

$$\begin{aligned}D_{AB}{}^{S0}(L_p)&=e^{-iA\varphi}d_{AB}{}^{S}(\theta)e^{iB(\varphi+i\alpha)}\,,\\ D_{AB}{}^{0S}(L_p)&=e^{-iA\varphi}d_{AB}{}^{S}(\theta)e^{iB(\varphi-i\alpha)}\,,\end{aligned} \tag{3.28}$$

with the angles (φ,θ,α) as in (2.3). It is of course necessary to express these angles as functions of the Mandelstam variables s and t consistently with our choice of frame.

4. POLES IN THE σ PLANE

A sufficient condition for the existence of the $SO(3,1)$ expansion employing only the principal series of *unitary* representations $D^{j_0\sigma}$ is that the amplitude as a function of energy s should fall faster than s^{-1} (the square integrability condition). For forward scattering, this condition is not satisfied. For the general case however, it would appear, purely empirically, that for a region of *large momentum transfer* $(-t)$ values, the amplitude is likely to decrease fast enough for the square-integrability criteria to be satisfied and the $SO(3,1)$ expansion in terms of only the principal series to be valid.

Let us fix t large and negative so that this is the case. Assume that the amplitude can be continued for complex values of σ and is meromorphic in a strip $|\mathrm{Re}\sigma|<L$. With Toller, if we now shift the integration path to the left, we may write[14] for large negative t

$$T=\int_{-i\infty}^{i\infty}d\sigma=\int_{-M-i\infty}^{-M+i\infty}+\sum\sigma\text{-plane pole terms}\,,$$

and then the pole contributions dominate asymptotically the contribution along the new shifted path. The poles lie on trajectories $\sigma=a_r(j_0,t)$ in the σ plane and clearly provide a generalization of Regge trajectories. The important point is that moving analytically along a trajectory from large $(-t)$ up to $t=0$, together with the assumption that $T\approx\sum$ (pole contributions), means that we shall also reach those values of t for which the amplitude was not square integrable in the first place. Thus the possibility of the formulas holding while we ride along the trajectory implies that no nonunitary representations need to be added to the $SO(3,1)$ expansion and all we require is the analytic continuation of the principal series representations. But before shifting the integration path arbitrarily to the left, it is important that we expand the amplitude into representation functions of the second kind $e^{j_0\sigma}$, in place of the $d^{j_0\sigma}$, since these have a well-defined asymptotic behavior.[15] More precisely, the e are defined by the split

$$d_{j'\mu j}{}^{j_0\sigma}(z)=e_{j'\mu j}{}^{j_0\sigma}(z)+\frac{\Gamma(j+\sigma+1)}{\Gamma(j-\sigma+1)}e_{j'\mu j}{}^{-j_0-\sigma}(z)\frac{\Gamma(j'-\sigma+1)}{\Gamma(j'+\sigma+1)}\,,$$

with the asymptotic property

$$e_{j'\mu j}{}^{j_0\sigma}(z)\sim z^{\sigma-1-|j_0-\mu|}\,.$$

Note the independence of this term from j and j'. The factorization hypothesis leads us to suppose that the pole structure is

$$T_{J'j'Jj}{}^{j_0\sigma}=\sum_r\frac{b_{J'j'r}(t)b_{Jjr}(t)}{\sigma-a_r(j_0,t)}\,,$$

where r labels the set of Toller poles. Introducing a single pole into the amplitude in (3.11) and retaining the leading term

$$\langle q'j'\mu|T(P^2)|qj\mu\rangle\sim b_{j'}(t)b_j(t)[j_0{}^2-a^2(j_0,t)]z^{-a(t)-1-|j_0-\mu|}\,,$$

where

$$z=\hat{q}\cdot\hat{q}'=2(s-u)/[(4M^2-t)(4M'^2-t)]^{1/2}.$$

It is the structure of the daughter-parent relationship which is important, and this has been worked out by Sciarrino and Toller[1] by expanding $SO(3,1)$ representations in terms of $SO(2,1)$ representations. The latter are based on the subgroup of $SO(3,1)$ which operates in the 023 subspace having states labeled by the two eigenvalues $J_{23}=m\bar{j}(\bar{j}+1)=J_{23}{}^2-J_{02}{}^2-J_{03}{}^2$. On the other hand, the $O(3)$ subgroup employed in the $SO(3,1)$ representations above use the eigenvalues $J_{12}=\mu$ and

[14] Because the $d^{j_0\sigma}$ functions satisfy the weak equivalence relation

$$d_{j\mu j'}{}^{-j_0-\sigma}=\frac{\Gamma(j+\sigma+1)}{\Gamma(j-\sigma+1)}d_{j\mu j'}{}^{j_0\sigma}\frac{\Gamma(j'-\sigma+1)}{\Gamma(j'+\sigma+1)},$$

the integration over σ defined in Sec. 3 over positive imaginary values can be extended over the entire imaginary axis provided also that we restrict the range j_0 to non-negative values up to $\min(j,j')$. Hereafter we adhere to this reinterpretation.

[15] J. Strathdee, J. F. Boyce, R. Delbourgo, and Abdus Salam, Internal Report, Trieste IC/67/9 (unpublished).

$j(j+1)=J_{12}^2+J_{23}^2+J_{31}^2$. We thus wish to pass from

$$\langle j_0\sigma j'\mu|e^{-i\zeta J_{03}}|j_0\sigma j\mu\rangle \quad\text{to}\quad \langle\tilde{j}m'|e^{-i\zeta J_{01}}|\tilde{j}m\rangle.$$

This connection is obtained from the identity

$$e^{-i\zeta J_{03}}=e^{\frac{1}{2}i\pi J_{31}}e^{-i\zeta J_{01}}e^{-\frac{1}{2}i\pi J_{31}}.$$

Therefore,

$$d_{j'\mu j}{}^{j_0\sigma}(\zeta)=\sum_{\nu\nu'} d_{\mu\nu'}{}^{j'}(-\tfrac{1}{2}\pi)\langle j'\nu'|e^{-i\zeta J_{01}}|j\nu\rangle d_{\nu\mu}{}^{j}(\tfrac{1}{2}\pi).$$

Once it is recognized that

$$\langle\tilde{j}m'=J_{23}|e^{-i\zeta J_{02}}|\tilde{j}m=J_{23}\rangle =\langle\tilde{j}m'=J_{12}|e^{-i\zeta J_{01}}|\tilde{j}m=J_{12}\rangle,$$

the introduction of a complete set of $O(2,1)$ states gives

$$d_{j'\mu j}{}^{j_0\sigma}(\zeta)=\sum_{\nu\nu'}\int_{\tilde{j}} d_{\mu\nu'}{}^{j'}(-\tfrac{1}{2}\pi) \times K_{\nu'}{}^{j_0\sigma}(\tilde{j},j')^* d_{\nu'\nu}{}^{\tilde{j}}(\zeta)K_{\nu}{}^{\tilde{j}_0\sigma}(\tilde{j},j)d_{\nu\mu}{}^{j}(\tfrac{1}{2}\pi),$$

where K denotes the overlap function from $O(3)$ to $O(2,1)$ base vectors and the integral over $\tilde{j}$ stands for the sum over the discrete and principal series of $O(2,1)$. K can be regarded as the expectation value of $e^{-\frac{1}{2}\pi J_{03}}$ and equals $d_{\tilde{j}\mu j}{}^{j_0\sigma}(\tfrac{1}{2}i\pi)$.

Consider the contribution of a single Toller pole to the amplitude

$$\langle j'\mu'|T|j\mu\rangle \sim b_{j'}b_j(j_0^2-a^2(j_0))d_{j'\mu j}{}^{j_0 a}(z)$$
$$=b_{j'}b_j(j_0^2-a^2)\sum_{\nu\nu'}\int_{\tilde{j}} d_{\mu'\nu'}{}^{j'}(-\tfrac{1}{2}\pi) \times K_{\nu'}{}^{j_0 a}(\tilde{j}j')^* d_{\nu'\nu}{}^{\tilde{j}}(z)K_{\nu}{}^{j_0 a}(\tilde{j},j)d_{\nu\mu}{}^{j}(\tfrac{1}{2}\pi).$$

The detailed properties of the $SO(3,1)\rightarrow SO(2,1)$ reduction formulas have been derived by Sciarrino and Toller.[1] The crux of the argument is the fact that the integrand K^*K as a function of $\tilde{j}$ exhibits poles at

$$\tilde{j}=\sigma-n-1 \quad (n=0,1,2).$$

This means that the Regge poles (identified with $\tilde{j}$) occur at the points

$$\alpha_n(t)=a(j_0,t)-1-n,$$

where n is the daughter number. The amplitude thereby decomposes as a sum of Regge terms.

$$\langle j'\mu|T|j\mu\rangle=b_{j'}b_j(j_0^2-a^2) \times\sum_{n\nu\nu'} d_{\mu\nu'}{}^{j}(-\tfrac{1}{2}\pi)d_{\nu'\nu}{}^{a-n-1}(z)d_{\nu\mu}{}^{j}(\tfrac{1}{2}\pi) \times[W_{j'\nu'}{}^{j_0 an*}K_{\nu'}{}^{j_0 an}(a-n-1,j)+W\leftrightarrow K],$$

where $W_{j\nu}{}^{j_0 an}$ is the residue of K at the daughter pole (n). It is evident that the residues of the daughters are completely determined in terms of the residue of the parent Toller poles and the universal factors W and K.

An important place where physical consequences of the scheme presented above might be looked for are the zeros of the amplitude associated[16] with e^{-j_0-a}. These occur at $a(t)=-\mu+1+n$ and $j'+1+n$ (for $j_0\leqslant\mu$). At those particular values of t the amplitude is expected to vanish.

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[16] The asymptotic form of $e_{j\mu j'}{}^{j_0\sigma}(z)$ is as follows:

$$e_{j\mu j'}{}^{j_0\sigma}(z)\sim[(2j+1)(2j'+1)]^{1/2} \times\left[\frac{\Gamma(j-\mu+1)\Gamma(j+j_0+1)\Gamma(j'-\mu+1)\Gamma(j'+j_0+1)}{\Gamma(j+\mu+1)\Gamma(j-j_0+1)\Gamma(j'+\mu+1)\Gamma(j'-j_0+1)}\right]^{1/2} \times\frac{(-)^{j'+\mu}\Gamma(\sigma+j'+1)\Gamma(-j_0-\sigma)}{\Gamma(j_0-\mu+1)\Gamma(\sigma-\mu+1)\Gamma(-\sigma+j'+1)}z^{-(\sigma+1+j_0-\mu)}, \quad j_0\geqslant\mu.$$