Lidiard's Theory of Itinerant-Electron Antiferromagnetism*

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Lidiard's theory of itinerant-electron antiferromagnetism is investigated in detail both with respect to the possible stable ground states and with respect to its spin-wave properties. As regards stability with respect to individual-particle excitations, saturated and unsaturated ferromagnetic states are possible over the full range of relative magnetizations. The antiferromagnetic state is stable against both collective and individual particle excitations only in the fully saturated case. A stable ferrimagnetic state is shown to exist when one spin direction is fully saturated and the other direction slightly less than fully saturated. The spin-wave dispersion law in the long-wave limit is given explicitly, derived from a Green's-function technique in random-phase approximation. The Lidiard model corresponds to a two-band itinerant model with both interband and intraband short-range exchange interactions, i.e., exchange integrals independent of wave vector.

I. INTRODUCTION

A TWO-BAND model of antiferromagnetism in metals based on a Stoner-type internal field[1,2] was proposed by Lidiard.[3] This internal-field model is in reality a short-range interaction model of a system of electrons in two bands having only one type of intra- and one type of interband exchange process treated in a generalized Hartree-Fock (HF) approximation.[2] In all the discussions of this model,[4,5] there is only a treatment of the properties of the antiferromagnetic ground state, and no discussion of the collective modes of excitations. Since the Lidiard model is mathematically the simplest band model of antiferromagnetism, a more detailed discussion of the ground-state properties than has been given previously seems worthwhile. We also present a calculation of the frequency- and wave-vector-dependent transverse spin susceptibility of the system employing a generalized random-phase approximation (RPA). Details of the calculation and comparison with other recently proposed models of itinerant electron antiferromagnetism are presented in a technical report now in preparation (Ref. 2).

II. ONE-PARTICLE HARTREE-FOCK GREEN'S FUNCTIONS

In Lidiard's model, no spin-density-wave–type interactions are envisaged, so a straightforward application of the result of Ref. 2 leads to the desired results. The assumptions made here are: In the limit of vanishing exchange interactions, the electrons occupy two bands. Because of exchange interactions among the electrons which are independent of wave vector, the two degenerate bands are separated into bands corresponding to the up and down spins. The quantization axes of the band polarizations are assumed to be the same and taken along the z axis. ζ_1, ζ_2 are the relative magnetizations of bands 1 and 2, respectively. To be quite general, ζ_1, ζ_2 may be allowed to vary between -1 and $+1$. Although the two bands may be similar in the dependence of energy on wave vector, they are assumed to have different spatial distributions so that different polarizations of the bands correspond to a spatially varying net spin. In particular, the two bands may be thought of as having wave functions with different amplitudes on the two antiferromagnetic sublattices. If the atoms of the two sublattices are identical, except for exchange the bands will be degenerate with each other and identical in shape.

We employ the following notations:

$\zeta_1=\zeta_2\neq\pm1$,	unsaturated ferromagnetic state (USF)
$\zeta_1=\zeta_2=\pm1$,	saturated ferromagnetic state (SF)
$\zeta_2=-\zeta_1\neq\pm1$,	unsaturated antiferromagnetic state (USAF)
$\zeta_2=-\zeta_1=\pm1$,	saturated antiferromagnetic state (SAF)

and $\zeta_1=1$, $0\geq\zeta_2\neq-\zeta_1$ (or $\zeta_1=1$, $\zeta_2\cong-\zeta_1$) and vice versa, ferrimagnetic state (F). In fact there, could be the whole

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range of ferrimagnetic unsaturation, $\zeta_1>0$, $\zeta_2\leq 0$, $|\zeta_2|<\zeta_1$. But only the case $\zeta_1=1$, $\zeta_2\cong-\zeta_1$ turns out to be interesting. The total number of electrons in the two bands N_1, N_2 are assumed equal $(=N)$.

The Hartree-Fock Green's functions are[2]

$$G_{\mathrm{HF}}(1k;\omega)=\begin{pmatrix}[\omega-E(1k)+\tfrac{1}{2}V_1N(1+\zeta_1)+\tfrac{1}{2}V_2N(1+\zeta_2)]^{-1} & 0\\ 0 & [\omega-E(1k)+\tfrac{1}{2}V_1N(1-\zeta_1)+\tfrac{1}{2}V_2N(1-\zeta_2)]^{-1}\end{pmatrix}, \tag{1a}$$

$$G_{\mathrm{HF}}(2k;\omega)=\begin{pmatrix}[\omega-E(2k)+\tfrac{1}{2}V_1N(1+\zeta_2)+\tfrac{1}{2}V_2N(1+\zeta_1)]^{-1} & 0\\ 0 & [\omega-E(2k)+\tfrac{1}{2}V_1N(1-\zeta_2)+\tfrac{1}{2}V_2N(1-\zeta_1)]^{-1}\end{pmatrix}. \tag{1b}$$

Here $\omega=(2n+1)\pi i/\beta$, where $\beta=1/kT$, is the usual energy variable. $E(1k)$, $E(2k)$ are the band energies. V_1 and V_2 are the intra- and interband exchange integrals, respectively. Lidiard assumed

$$E(1k)=E(2k)=k^2/2m, \tag{2}$$

i.e., the two bands are doubly degenerate in the entire k space, another assumption of the model not previously noted explicitly, but which follows, as stated above, if the sublattice atoms are identical except for exchange effects.

Furthermore we let

$$mV_1k_F/2\pi^2=\tfrac{3}{2}(K\theta_1/\epsilon_F),\qquad mV_2k_F/2\pi^2=\tfrac{3}{2}(K\theta_2/\epsilon_F) \tag{3}$$

or equivalently $NV_1=2K\theta_1$, $NV_2=2K\theta_2$. Here k_F is the paramagnetic Fermi momentum. We use units in which $\hbar=1$. The poles of G in the ω plane in (1a) and (1b) give the single-particle energies, which were assumed by Lidiard. (Lidiard takes $\zeta_2\to-\zeta_1$ right from the start, so that his condition for antiferromagnetism is $\zeta_2=\zeta_1=1$).

The above equations follow if we truncate the complete Hamiltonian of an interacting Bloch electron system to the form

$$\mathcal{H}=\sum_{lk\sigma}E(lk)\,a_{lk\sigma}{}^{\dagger}a_{lk\sigma}+\tfrac{1}{2}V_1\sum_{l_1}\sum_{k_1k_2q}\sum_{\sigma_1\sigma_2}a^{\dagger}{}_{l_1k_1+q\sigma_1}a^{\dagger}{}_{l_1k_2-q\sigma_2}a_{l_1k_2\sigma_2}a_{l_1k_1\sigma_1}+\tfrac{1}{2}V_2\sum_{(l_1\neq l_2)}\sum_{k_1k_2q}\sum_{\sigma_1\sigma_2}a^{\dagger}{}_{l_1k_1+q\sigma_1}a^{\dagger}{}_{l_2k_2-q\sigma_2}a^{l}{}_{1k_2\sigma_2}a_{l_2k_1\sigma_1}, \tag{4}$$

employ the usual HF approximation, and neglect the direct Hartree term which is assumed to be cancelled by the background. Here l stands for band index, σ for spin. $E(lk)$ are the band energies of the noninteracting electrons.

III. THE GROUND-STATE ENERGY OF THE SYSTEM

The total energy of the system in HF per particle per unit volume in units of ϵ_F, the paramagnetic Fermi energy, is then

$$\begin{aligned}\mathcal{E}(\zeta_1,\zeta_2)=\langle\mathcal{H}\rangle/N\Omega_0\epsilon_F=\tfrac{3}{10}[(1+\zeta_1)^{5/3}+(1-\zeta_1)^{5/3}+(1+\zeta_2)^{5/3}+(1-\zeta_2)^{5/3}]\\ -\tfrac{1}{2}(K\theta_1/\epsilon_F)(2+\zeta_1^2+\zeta_2^2)-(K\theta_2/\epsilon_F)(1+\zeta_1\zeta_2);\qquad(-1\leq(\zeta_1,\zeta_2)\leq 1).\end{aligned} \tag{5}$$

Here the first term on the right is the average kinetic energy of the electrons in the two bands, the second term is the intraband exchange energy, and the last term is the interband exchange energy. The *absolute minimum* of this as a function of the two variables ζ_1, ζ_2 in the domain $-1\leq\zeta_1\leq 1$, $-1\leq\zeta_2\leq 1$ is the ground state of the system. To find the absolute minimum, we must proceed in three steps. We first discuss the *interior parts of the square*, $-1<(\zeta_1,\zeta_2)<1$. The critical points inside the square are given by

$$\partial\mathcal{E}/\partial\zeta_1=0\quad\text{and}\quad\partial\mathcal{E}/\partial\zeta_2=0. \tag{6}$$

Since a Taylor's series exists for $\mathcal{E}$ everywhere inside the square, the conditions for minimum are

$$(\partial^2\mathcal{E}/\partial\zeta_1^2)(\partial^2\mathcal{E}/\partial\zeta_2^2)-(\partial^2\mathcal{E}/\partial\zeta_1\partial\zeta_2)^2\geq 0,\qquad \partial^2\mathcal{E}/\partial\zeta_1^2>0,\qquad \partial^2\mathcal{E}/\partial\zeta_2^2>0. \tag{7}$$

Next we discuss the points on the sides of the square. Since $\mathcal{E}$ is symmetrical under the transformation $(\zeta_1,\zeta_2)\to(\zeta_2,\zeta_1)$, $(\zeta_1,\zeta_2)\to(-\zeta_1,-\zeta_2)$ it is sufficient to examine the existence of a minimum on the lines $(\zeta_1,+1)$, $(\zeta_1,-1)$, with $-1<\zeta_1<1$. We must make sure that the derivative of $\mathcal{E}$ towards the interior of the square is positive as we move away from the boundary. Lastly, we must compare the energies at the corners of the square, with those energies at the minima in the interior and on the boundary to decide the absolute minimum of $\mathcal{E}$. Again, one must ensure that the directional derivatives of $\mathcal{E}$ at the corners towards the interior of the allowed region are positive. Here again, it is sufficient to examine the points (1, 1) and (1, −1). We give below the main results of the calculations described above.

A. Interior Points

The conditions $\partial\mathcal{E}/\partial\zeta_1=0=\partial\mathcal{E}/\partial\zeta_2$ determine $(K\theta_1/\epsilon_F)$ and $(K\theta_2/\epsilon_F)$:

$$\left.\begin{aligned}(K\theta_1/\epsilon_F)&=\tfrac{1}{2}[\zeta_1 f(\zeta_1)-\zeta_2 f(\zeta_2)]/(\zeta_1^2-\zeta_2^2),\\(K\theta_2/\epsilon_F)&=\tfrac{1}{2}[\zeta_1 f(\zeta_2)-\zeta_2 f(\zeta_1)]/(\zeta_1^2-\zeta_2^2),\end{aligned}\right\}\quad(\zeta_1^2\neq\zeta_2^2) \tag{8a}$$

$$f(\zeta)=[(1+\zeta)^{2/3}-(1-\zeta)^{2/3}],$$

$$\partial^2\mathcal{E}/\partial\zeta_1^2=\tfrac{1}{3}[(1+\zeta_1)^{-1/3}+(1-\zeta_1)^{-1/3}]-(K\theta_1/\epsilon_F), \tag{8b}$$

$$\partial^2\mathcal{E}/\partial\zeta_2^2=\tfrac{1}{3}[(1+\zeta_2)^{-1/3}+(1-\zeta_2)^{-1/3}]-(K\theta_1/\epsilon_F), \tag{8c}$$

$$\partial^2\mathcal{E}/\partial\zeta_1\partial\zeta_2=-(K\theta_2/\epsilon_F). \tag{8d}$$

By a direct numerical calculation it may be shown that $(\partial^2\mathcal{E}/\partial\zeta_1^2)(\partial^2\mathcal{E}/\partial\zeta_2^2)-(\partial^2\mathcal{E}/\partial\zeta_1\partial\zeta_2)^2<0$ in the entire ζ_1, ζ_2 plane except along the main diagonals $\zeta_2=\pm\zeta_1$, where it is zero. We may substantiate this analytically by verifying this statement for ζ_1, ζ_2 small compared to unity in absolute magnitude. For small ζ_1, ζ_2, Eqs. (8) reduce to

$$(K\theta_1/\epsilon_F)\sim\tfrac{2}{3}+(4/81)(\zeta_1^2+\zeta_2^2),$$

$$(K\theta_2/\epsilon_F)\sim-(4/81)\zeta_1\zeta_2,$$

$$\partial^2\mathcal{E}/\partial\zeta_1^2\sim(4/81)(2\zeta_1^2-\zeta_2^2),\qquad\partial^2\mathcal{E}/\partial\zeta_2^2\sim(4/81)(2\zeta_2^2-\zeta_1^2),\qquad\partial^2\mathcal{E}/\partial\zeta_1\partial\zeta_2\sim(4/81)\zeta_1\zeta_2, \tag{9}$$

and

$$(\partial^2\mathcal{E}/\partial\zeta_1^2)(\partial^2\mathcal{E}/\partial\zeta_2^2)-(\partial^2\mathcal{E}/\partial\zeta_1\partial\zeta_2)^2\sim-2(4/81)(\zeta_1^2-\zeta_2^2)^2. \tag{10}$$

Since (10) vanishes only when $\zeta_1^2=\zeta_2^2$ and both second derivatives are positive under these conditions, it follows that minima in the interior region are found only on the main diagonals. The same conclusion may be verified numerically for arbitrary values of ζ_1 and ζ_2.

Along the main diagonals we have for $\zeta_2\to\zeta_1$(USF) and $\zeta_2\to-\zeta_1$(USAF)

$$(K\theta_1/\epsilon_F)_{\text{USF}}=(1/4\zeta_1)f(\zeta_1)+\tfrac{1}{6}[(1+\zeta_1)^{-1/3}+(1-\zeta_1)^{-1/3}],$$

$$(K\theta_2/\epsilon_F)_{\text{USF}}=(1/4\zeta_1)f(\zeta_1)-\tfrac{1}{6}[(1+\zeta_1)^{-1/3}+(1-\zeta_1)^{-1/3}],$$

$$(\partial^2\mathcal{E}/\partial\zeta_1^2)_{\text{USF}}=(\partial^2\mathcal{E}/\partial\zeta_2^2)_{\text{USF}}=(\partial^2\mathcal{E}/\partial\zeta_1\partial\zeta_2)_{\text{USF}}=-(K\theta_2/\epsilon_F)_{\text{USF}}. \tag{11}$$

Similarly,

$$(K\theta_1/\mathcal{E}_{\text{F}})_{\text{USAF}}=(K\theta_1/\epsilon_F)_{\text{USF}},\qquad(K\theta_2/\epsilon_F)_{\text{USAF}}=-(K\theta_2/\epsilon_F)_{\text{USF}},$$

$$(\partial^2\mathcal{E}/\partial\zeta^2)_{\text{USAF}}=(\partial^2\mathcal{E}/\partial\zeta_2^2)_{\text{USAF}}=-(\partial^2\mathcal{E}/\partial\zeta_1\partial\zeta_2)_{\text{USAF}}=(K\theta_2/\epsilon_F)_{\text{USAF}}. \tag{12}$$

Thus, the only *allowed* states in the interior of the square are USF and USAF provided

$$(K\theta_2/\epsilon_F)_{\text{USF}}<0,\qquad(K\theta_2/\epsilon_F)_{\text{USAF}}>0. \tag{13}$$

From (11), for small ζ_1, $(K\theta_2/\epsilon_F)_{\text{USAF}}>0$ and from (12), we see that USF and USAF may indeed be *minima*. The same conclusion may be verified numerically for large values of ζ_1.

B. Boundaries

We will do both the cases $(\zeta_1,\pm1)$ together and the results for $(\pm1,\zeta_2)$ are the same as these.

$$\mathcal{E}(\zeta_1,\pm1)=\tfrac{3}{10}\{2^{5/3}+(1+\zeta_1)^{5/3}+(1-\zeta_1)^{5/3}\}-\tfrac{1}{2}(K\theta_1/\epsilon_F)(3+\zeta_1^2)-(K\theta_2/\epsilon_F)(1\pm\zeta_1),\qquad(-1<\zeta_1<1). \tag{14a}$$

So,

$$\partial\mathcal{E}/\partial\zeta_1=0=\tfrac{1}{2}f(\zeta_1)-(K\theta_1/\epsilon_F)\zeta_1\pm(K\theta_2/\epsilon_F), \tag{14b}$$

$$\partial^2\mathcal{E}/\partial\zeta_1^2>0\quad\text{or}\quad(K\theta_1/\epsilon_F)<\tfrac{1}{3}[(1+\zeta_1)^{-1/3}+(1-\zeta_1)^{-1/3}]. \tag{14c}$$

By considering the directional derivative of $\mathcal{E}(\zeta_1, \zeta_2)$ into the interior from these lines, we arrive at

$$(K\theta_1/\epsilon_F) \pm (K\theta_2/\epsilon_F)\zeta_1 > 2^{-1/3}. \tag{14d}$$

Thus the set of conditions for the points on the boundary $(\zeta_1, \pm 1)$ to be minima are

$$(K\theta_2/\epsilon_F) = \mp (K\theta_1/\epsilon_F)\zeta_1 \pm \tfrac{1}{2}f(\zeta_1),$$

$$[2^{-1/3} - \tfrac{1}{2}\zeta_1 f(\zeta_1)]/(1-\zeta_1^2) < (K\theta_1/\epsilon_F) < \tfrac{1}{3}[(1+\zeta_1)^{-1/3} + (1-\zeta_1)^{-1/3}]. \tag{15a}$$

If we take $\eta = (1-\zeta_1)$, as small in the inequality of (15a), we have to lowest order in η,

$$\tfrac{1}{3}2^{2/3} + \tfrac{1}{4}\eta^{-1/3} < (K\theta_1/\epsilon_F) < \tfrac{1}{6}2^{2/3} + \tfrac{1}{3}\eta^{-1/3}. \tag{15b}$$

These inequalities are thus compatible only if

$$\eta < \tfrac{1}{32} \quad \text{or} \quad \zeta_1 > \tfrac{31}{32}. \tag{15c}$$

It is readily verified by direct numerical calculation that the inequalities (15a) are incompatible except when $|\zeta_1| > \frac{31}{32}$. Note that the functions on both sides of the inequalities in (15a) are even in ζ_1. This implies that only a small portion of the points on the boundary near the corners can be candidates for the ground state of the system when (15a) is satisfied. These are nearly compensated ferrimagnetic states (F) when ζ_1 is positive, $\zeta_2 = -1$ and ζ_1 is negative, $\zeta_2 = +1$.

C. Corners

To decide the absolute minimum, we compare the corner energies with each other, with those on the main diagonal, and with those on the boundary lines near the corners. We must also make sure that $\partial\mathcal{E}/\partial n$ towards the interior of the square is positive, away from the corner. Again, it is sufficient to consider the two corners, (1, 1) and (1, −1).

(a) $$\mathcal{E}_{\rm SF}(1,1) = \tfrac{3}{5}2^{2/5} - 2(K\theta_1/\epsilon_F) - 2(K\theta_2/\epsilon_F), \tag{16}$$

$$(\partial\mathcal{E}/\partial n)_{(1\,1)} > 0 \quad \text{gives} \quad (K\theta_1/\epsilon_F) + (K\theta_2/\epsilon_F) > 2^{-1/3}. \tag{17}$$

(b) $$\mathcal{E}_{\rm SAF}(1,-1) = \tfrac{3}{5}2^{5/3} - 2(K\theta_1/\epsilon_F), \tag{18}$$

and

$$(\partial\mathcal{E}/\partial n)_{(1-1)} > 0 \quad \text{gives} \quad (K\theta_1/\epsilon_F) - (K\theta_2/\epsilon_F) > 2^{-1/3}. \tag{19}$$

From (16) and (18),

$$\mathcal{E}_{\rm SAF}(1,-1) - \mathcal{E}_{\rm SF}(1,1) = 2(K\theta_2/\epsilon_F). \tag{20}$$

Therefore, SAF will be lower than SF if

$$(K\theta_2/\epsilon_F) < 0. \tag{21}$$

Now from (13), USAF will be a ground state if $(K\theta_2/\epsilon_F) > 0$. Hence under these conditions, SAF will be lower than USAF, because if (19) is satisfied, there is no allowable ζ_1 which leads to the same $K(\theta_1-\theta_2)/\epsilon_F$ satisfying the condition of a minimum for USAF. This follows from (12) and (11) that

$$(K(\theta_1-\theta_2)/\epsilon_F)_{\rm USAF} = (1/2\zeta_1)f(\zeta_1) < 2^{-1/3} \quad \text{for} \quad |\zeta_1| < 1.$$

Thus, we have proved that USF, USAF, and F can be relative minima, and under certain conditions SF and SAF can be absolute minima.

IV. THE SPIN-WAVE EXCITATIONS OF THE SYSTEM

We now compute the transverse spin susceptibility $\chi_{+-}(q,\omega)$ by a method outlined in Ref. 2 in RPA. We obtain

$$\chi_{+-}(q,\omega) = -\frac{C_1 + C_2 - 2(V_1 - V_2)C_1C_2}{1 - V_1(C_1+C_2) + (V_1^2 - V_2^2)C_1C_2}, \tag{22}$$

where

$$C_1 = i\int \frac{d^4\bar{q}}{(2\pi)^4} G_{\downarrow\downarrow}(1q+\bar{q})G_{\uparrow\uparrow}(1\bar{q}),$$

$$C_2 = i\int \frac{d^4\bar{q}}{(2\pi)^4} G_{\downarrow\downarrow}(2q+\bar{q})G_{\uparrow\uparrow}(2\bar{q}). \tag{23}$$

Here the G's are the Green's functions given by 1(a) and (1b). This can also be derived by an equation-of-motion method in RPA from the Hamiltonian (4), given in Sec. II. We evaluate the pole of χ_{+-} in the limit $\omega, q \to 0$ using[2]

$$C_1 \sim -\frac{N\zeta_1}{\bar{\Omega}_1^3}\left[\bar{\Omega}_1^2 + \frac{q^2}{2m\zeta_1}\bar{\Omega}_1 + \frac{q^2}{2m\zeta_1}A(\zeta_1)\right],$$

$$C_2 \sim -\frac{N\zeta_2}{\bar{\Omega}_2^3}\left[\bar{\Omega}_2^2 + \frac{q^2}{2m\zeta_2}\bar{\Omega}_2 + \frac{q^2}{2m\zeta_2}A(\zeta_2)\right], \tag{24}$$

where

$$\bar{\Omega}_1 = \omega - a, \qquad \bar{\Omega}_2 = \omega - b, \qquad a = 2K\theta_1\zeta_1 + 2K\theta_2\zeta_2,$$

$$b = 2K\theta_1\zeta_2 + 2K\theta_2\zeta_1,$$

and

$$A(\zeta) = \tfrac{3}{5}\epsilon_F[(1+\zeta)^{5/3} - (1-\zeta)^{5/3}]. \tag{25}$$

After some algebra we find the poles at

$$\omega^2[ab + 4(a+b)(\zeta_1+\zeta_2)K\theta_2] - 2\omega ab(\zeta_1+\zeta_2)K\theta_2 + (q^2/2m)(2K\theta_2)[2ab - A(\zeta_1)b - A(\zeta_2)a] = 0. \tag{26}$$

We must stipulate that the spin-wave frequency is positive and real[5] for stability of the system. From the discussion of the ground states, we must consider the spin waves for USF, USAF, SF, SAF, and F states.

For $\zeta_2=\zeta_1$, the ω^2 term in (26) is neglected as o higher order, so that, since in that case, $a=b=2(K\theta_1+K\theta_2)\zeta_1$, $A(\zeta_1)=A(\zeta_2)$,

$$\omega\sim\frac{q^2}{2m\zeta_1}\left[1-\frac{1}{5}\left\{\frac{(1+\zeta_1)^{5/3}-(1-\zeta_1)^{5/3}}{[K(\theta_1+\theta_2)/\epsilon_F]\zeta_1}\right\}\right]. \quad (27)$$

This is just the spin-wave dispersion in the USF and SF found already in Ref. 2 for spin waves in the short-range model. We note that inter- and intraband exchange enter only in terms of their sum, $K(\theta_1+\theta_2)/\epsilon_F$.

For $\zeta_2=-\zeta_1$, however, the linear term in ω *vanishes identically* and hence the first nonvanishing term in ω is the ω^2 term. Then $b=-a=-2(K\theta_1-K\theta_2)\zeta_1$, $A(\zeta_2)=-A(\zeta_1)$, and so

$$\omega^2\sim\left(\frac{qk_F}{m}\right)^2\left(\frac{K\theta_2}{\epsilon_F}\right)\left[-1+\frac{1}{5}\left\{\frac{(1+\zeta_1)^{5/3}-(1-\zeta_1)^{5/3}}{[K(\theta_1-\theta_2)/\epsilon_F]\zeta_1}\right\}\right]. \quad (28)$$

This correctly predicts a dispersion relation which is linear in the wave vector q, which is a well-known result of other theories of AF states. Now, for USAF, $K\theta_2/\epsilon_F>0$, and $K(\theta_1-\theta_2)/\epsilon_F=(1/2\zeta_1)f(\zeta_1)$ from the ground-state stability conditions. Therefore,

$$\omega^2_{\rm USAF}\sim\left(\frac{qk_F}{m}\right)^2\left(\frac{K\theta_2}{\epsilon_F}\right)\left[-1+\frac{2}{5}\left\{\frac{(1+\zeta_1)^{5/3}-(1-\zeta_1)^{5/3}}{(1+\zeta_1)^{2/3}-(1-\zeta_1)^{2/3}}\right\}\right]. \quad (29)$$

The quantity in square brackets is always negative, as can be verified by direct numerical calculation for all ζ_1, and can also be verified algebraically for small ζ_1, where the quantity in square brackets reduces to $-\frac{1}{9}\zeta_1^2$. Hence, the long-wavelength spin waves are *unstable* for USAF. Thus USAF cannot be a ground state of the system at all, because although it is stable against individual particle excitations, it becomes unstable against long-wave collective oscillations. This is in disagreement with Lidiard.[1] The SAF state may be a proper ground state, however, since in this case $K\theta_2/\epsilon_F<0$. Hence $\omega_{\rm SAF}^2$ is positive [the square bracket in (29) in this case is $-\frac{1}{5}$] and the SAF spin waves are stable. For the F state, $\zeta_1=1$, $\zeta_2\sim-1$ and so $(\zeta_1+\zeta_2)\sim 0$ in (26) and hence $A(\zeta_1)\sim-A(\zeta_2)$,

$$\omega_{\rm F}^2\sim(qk_F/m)^2(-K\theta_2/\epsilon_F)[1-(A(\zeta_1)/2ab)(b-a)]. \quad (30)$$

From (15a), $(K\theta_2/\epsilon_F)$ can be negative, and hence the spin waves in the F state have the same form of dispersion as SAF states. The result that such a model exhibits only nearly saturated antiferromagnetism is reasonable if we recall the discussion of Bloch states of this model given in Sec. II. The requirements that the Bloch functions for the two identical bands (so far as dispersion law is concerned) be mutually orthogonal suggests that they cannot have much overlap, and hence cannot differ much from full saturation. It may be of interest to mention that by analogy with the Heisenberg (localized) model of magnetism, the results obtained here concerning the spin-wave dispersion relations may be connected with the ideas of broken symmetry of the ground states of these systems and their respective collective excitations.[6] $\omega\cong q$ in the F state because of the smallness of ζ; the broken-symmetry arguments of Anderson[6] for the AF state seem to hold in the cases (AF and F) considered here also.

V. CONCLUSIONS

We have shown, by an analysis of both minimization of energy and the stability of long wavelength spin waves, that the Lidiard[3] model of antiferromagnetism is capable of explaining only the fully saturated antiferromagnetism and also a ferrimagnetic state near saturation. A physical origin of this result is also given. The itinerant antiferromagnetic spin-wave dispersion is very similar in structure to the well-known result of other theories of AF states.

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