

pressure for the metal Hg, as quoted in Ref. 4. However, as shown in Fig. 2, a much better prediction for vaporization entropy with this nonzero $\phi_1(T)$ is noted for the metals than for $\phi(T) \equiv 0$.

In conclusion, it should be noted that recent statistical mechanical treatments for a condensing fluid derive not only the van der Waals state equation, but also the Maxwell equal-area rule.⁵⁻⁷ Thus, there seems to be some difference between these statistical treat-

ments and the thermodynamic arguments given here. A resolution of this difference appears desirable.

These efforts are motivated by the need for a simple yet fairly accurate model for describing the state and dynamical behavior of superheated liquid metals over large temperature ranges, such as are encountered in exploding wire experiments.⁸

The assistance of D. C. Mylin and F. H. McIntosh in providing the numerical analysis is gratefully acknowledged.

⁴ A. V. Grosse, J. Inorg. Nucl. Chem. **22**, 23 (1961).

⁵ M. Kac, G. E. Uhlenbeck, and P. C. Hemmer, J. Math. Phys. **4**, 216 (1963); **4**, 229 (1963).

⁶ N. G. van Kampen, Phys. Rev. **135**, A362 (1964).

⁷ J. L. Lebowitz and O. Penrose, J. Math. Phys. **7**, 98 (1966).

⁸ F. D. Bennett, G. D. Kahl, and E. H. Wedemeyer, in *Exploding Wires*, edited by W. G. Chace and H. K. Moore (Plenum Press, Inc., New York, 1964), Vol. 3, p. 65.

Long-Range Order in Superfluid Helium*

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Fluctuations in the phase of the condensate wave function appearing in the single-particle Green's function for a condensed Bose system are examined in one, two, and three dimensions. The results indicate that at nonzero temperatures, long-range order will appear only in three dimensions.

I. INTRODUCTION

THE appearance of a condensate in He^4 is accompanied by coherent, large-scale fluctuations in the phase of the condensate wave function. This phase coherence is responsible for the long-range order in the system. Physically this means that particles, far apart in space, are intimately correlated. Yang¹ has shown that these long-range correlations are manifest in the asymptotic behavior (for large spacial separations) of appropriate correlation functions of the system. In particular, these correlation functions, instead of vanishing as in a normal system, approach a nonzero constant. In He^4 the correlation function of interest is the single-particle Green's function, while in a superconductor it is the two-particle Green's function describing paired states. Rice² has argued that a superconducting tran-

sition will not occur in one or two dimensions. It is the purpose of this work to show that similar conclusions may apply to the He^4 system.

The behavior of the equilibrium, single-particle Green's function, $G(\mathbf{r}, t; \mathbf{r}', t)$, as $|\mathbf{r} - \mathbf{r}'| \rightarrow \infty$, is determined entirely by the long-wavelength correlations. Information about these long-wavelength correlations is the content of the hydrodynamic equations. Kadanoff and Martin³ have shown that one may obtain the long-wavelength, low-frequency correlation functions, which describe the linear response of a system to a disturbance, from the hydrodynamic equations. These response functions contain all of the information inherent in the linearized hydrodynamic equations.

The hydrodynamic equations for superfluid helium are well known, they are just the two-fluid equations of Tisza⁴ and Landau.⁵ The corresponding response functions have been obtained by Hohenberg and Martin.⁶

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¹ C. N. Yang, Rev. Mod. Phys. **34**, 694 (1962).

² T. M. Rice [Phys. Rev. **140**, A1889 (1965)] reached these conclusions by using the Landau-Ginzburg molecular-field approximation for the free energy.

³ L. P. Kadanoff and P. C. Martin, Ann. Phys. (N. Y.) **24**, 419 (1963).

⁴ L. Tisza, Nature **141**, 913 (1938).

⁵ L. Landau, J. Phys. USSR **5**, 71 (1941). A translation may be found in I. M. Khalatnikov, *Introduction to the Theory of Superfluidity* (W. A. Benjamin, Inc., New York, 1965).

⁶ P. C. Hohenberg and P. C. Martin, Phys. Rev. Letters **12**, 69 (1964); Ann. Phys. (N. Y.) **34**, 291 (1965).

We shall use these response functions to investigate the asymptotic behavior of the single-particle Green's function in one, two, and three dimensions.

II. THE CORRELATION FUNCTIONS

Penrose and Onsager⁷ showed that the single-particle density matrix in superfluid helium can be split into two parts. For large spacial separations the normal part vanishes and only the contributions from the condensed state remain. Thus, in equilibrium, the single-particle Green's function has the form

$$G(\mathbf{r}, t; \mathbf{r}', t') \xrightarrow{|\mathbf{r}-\mathbf{r}'| \rightarrow \infty} -i \langle [n_0(\mathbf{r}, t)]^{1/2} e^{i\varphi(\mathbf{r}, t)} \times [n_0(\mathbf{r}', t')]^{1/2} e^{-i\varphi(\mathbf{r}', t')} \rangle, \quad (1)$$

where $n_0(\mathbf{r}, t)$ is the local condensate density and $\varphi(\mathbf{r}, t)$ is the phase associated with the wave function. To lowest order fluctuations in the condensate density do not affect the asymptotic behavior of G .⁸ However, long-wavelength fluctuations will appear in the condensate phase,⁹ so we need only consider

$$G(\mathbf{r}, \mathbf{r}'; t) \xrightarrow{|\mathbf{r}-\mathbf{r}'| \rightarrow \infty} -in_0 \langle e^{i[\varphi(\mathbf{r}, t) - \varphi(\mathbf{r}', t)]} \rangle. \quad (2)$$

It is convenient to develop $\langle \exp i[\varphi(\mathbf{r}, t) - \varphi(\mathbf{r}', t)] \rangle$ in a cumulant expansion.¹⁰ Denoting

$$X \equiv i[\varphi(\mathbf{r}, t) - \varphi(\mathbf{r}', t)],$$

then

$$\langle e^X \rangle = \exp \left\{ \sum_{\xi=1}^{\infty} \frac{1}{\xi!} \langle X^\xi \rangle_c \right\}, \quad (3)$$

where $\langle \rangle_c$ denotes the cumulant average.

The first term in the exponent of (3) is just

$$\langle X \rangle_c = \langle X \rangle = i \langle \varphi(\mathbf{r}, t) - \varphi(\mathbf{r}', t) \rangle.$$

This is zero since the thermodynamic average of the phase is independent of space and time in equilibrium. The second term is more interesting;

$$\begin{aligned} \frac{1}{2} \langle X^2 \rangle_c &\equiv \frac{1}{2} \{ \langle X^2 \rangle - \langle X \rangle^2 \} \\ &= -\frac{1}{2} \langle [\varphi(\mathbf{r}, t) - \varphi(\mathbf{r}', t)]^2 \rangle. \end{aligned} \quad (4)$$

We can evaluate (4) explicitly in the long-wavelength limit.

We define the n -dimensional Fourier transform of $\varphi(\mathbf{r}, t)$ ¹¹

$$\varphi(\mathbf{r}, t) = \Omega_n \int \frac{d^n k}{(2\pi)^n} e^{i\mathbf{k} \cdot \mathbf{r}} \varphi(\mathbf{k}, t),$$

where Ω_n is the volume of the n -dimensional system. Then

$$\begin{aligned} &-\frac{1}{2} \langle [\varphi(\mathbf{r}, t) - \varphi(\mathbf{r}', t)]^2 \rangle \\ &= -\frac{1}{2} \Omega_n^2 \int \frac{d^n k}{(2\pi)^n} \int \frac{d^n k'}{(2\pi)^n} \langle \varphi(\mathbf{k}, t) \varphi(\mathbf{k}', t) \rangle e^{i(\mathbf{k}+\mathbf{k}') \cdot \mathbf{r}} \\ &\quad \times [1 - e^{-i\mathbf{k} \cdot (\mathbf{r}-\mathbf{r}')} - e^{-i\mathbf{k}' \cdot (\mathbf{r}-\mathbf{r}')} + e^{-i(\mathbf{k}+\mathbf{k}') \cdot (\mathbf{r}-\mathbf{r}')}]. \end{aligned} \quad (5)$$

In equilibrium this can only be a function of $|\mathbf{r}-\mathbf{r}'|$, so $\langle \varphi(\mathbf{k}, t) \varphi(\mathbf{k}', t) \rangle$ must be zero unless $\mathbf{k}' = -\mathbf{k}$. Also, the gradient of the phase is the superfluid velocity,^{6,12} so $i\mathbf{k}\varphi(\mathbf{k}, t) = \mathbf{v}_s(\mathbf{k}, t)$, and (5) reduces to

$$\begin{aligned} &-\frac{1}{2} \langle [\varphi(\mathbf{r}, t) - \varphi(\mathbf{r}', t)]^2 \rangle \\ &= -\Omega_n \int \frac{d^n k}{(2\pi)^n} \frac{1}{k^2} \langle \mathbf{v}_s(\mathbf{k}, t) \mathbf{v}_s(-\mathbf{k}, t) \\ &\quad + \mathbf{v}_s(-\mathbf{k}, t) \mathbf{v}_s(\mathbf{k}, t) \rangle \sin^2 \frac{\mathbf{k} \cdot (\mathbf{r}-\mathbf{r}')}{2}. \end{aligned} \quad (6)$$

We shall use the results of Hohenberg and Martin⁶ to evaluate the correlation function appearing in (6).

The imaginary part of the superfluid velocity response function $\chi_{\mathbf{v}_s \mathbf{v}_s}(\mathbf{k}, \omega) = \chi'_{\mathbf{v}_s \mathbf{v}_s}(\mathbf{k}, \omega) + i\chi''_{\mathbf{v}_s \mathbf{v}_s}(\mathbf{k}, \omega)$ is defined by¹³

$$\begin{aligned} \frac{1}{2} \langle [\mathbf{v}_s(\mathbf{r}, t), \mathbf{v}_s(\mathbf{r}', t')] \rangle &= \Omega_n \int \frac{d^n k}{(2\pi)^n} \int \frac{d\omega}{2\pi} \chi''_{\mathbf{v}_s \mathbf{v}_s}(\mathbf{k}, \omega) \\ &\quad \times e^{i\mathbf{k} \cdot (\mathbf{r}-\mathbf{r}') - i\omega(t-t')}, \end{aligned} \quad (7)$$

where $[A, B] = AB - BA$. This may be rewritten in terms of the anticommutator by using the fluctuation-dissipation theorem⁶:

$$\begin{aligned} &\int d(t-t') e^{i\omega(t-t')} \langle \{ \mathbf{v}_s(\mathbf{r}, t), \mathbf{v}_s(\mathbf{r}', t') \} \rangle \\ &= \coth \frac{\beta\omega}{2} \int d(t-t') e^{i\omega(t-t')} \langle [\mathbf{v}_s(\mathbf{r}, t), \mathbf{v}_s(\mathbf{r}', t')] \rangle, \end{aligned}$$

where $\{A, B\} = AB + BA$, and β is the inverse temperature in energy units. Thus in equilibrium

$$\begin{aligned} &\int \frac{d\omega}{2\pi} \coth \frac{\beta\omega}{2} \chi''_{\mathbf{v}_s \mathbf{v}_s}(\mathbf{k}, \omega) \\ &= \frac{1}{2} \langle \mathbf{v}_s(\mathbf{k}, t) \mathbf{v}_s(-\mathbf{k}, t) + \mathbf{v}_s(-\mathbf{k}, t) \mathbf{v}_s(\mathbf{k}, t) \rangle. \end{aligned} \quad (8)$$

⁷ O. Penrose and L. Onsager, Phys. Rev. **104**, 576 (1956).

⁸ J. W. Kane and L. P. Kadanoff, J. Math. Phys. **6**, 1902 (1965).

⁹ The analogous point has been made for superconductors by B. Ambegaokar and L. P. Kadanoff, Nuovo Cimento **22**, 914 (1961).

¹⁰ R. Kubo, J. Phys. Soc. Japan **17**, 1100 (1962).

¹¹ We use units in which $\hbar = 1$.

¹² N. Bogoliubov, *Hydrodynamics of the Superfluid Liquid* (Dubna, USSR, 1963).

¹³ This definition of the Fourier transform differs from that of Hohenberg and Martin by a factor of the volume.

Using (8) in (6) we obtain

$$\frac{1}{2}\langle[\varphi(\mathbf{r},t)-\varphi(\mathbf{r}',t)]^2\rangle=2\Omega_n\int\frac{d^n k}{(2\pi)^n}\int\frac{d\omega}{2\pi}\coth\frac{\beta\omega}{2}\times\chi''_{\mathbf{v}_s\mathbf{v}_s}(\mathbf{k},\omega)\frac{1}{k^2}\sin^2\frac{\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}{2}. \quad (9)$$

We are looking for divergences in the fluctuations given by (9), divergences which we expect² to emerge from the long-wavelength part of the integrand. Since (8) is positive definite, we can be sure that there are no canceling divergences in (9). Consequently, we can concentrate on the long-wavelength part of this integral by inserting a cutoff at $k=Q$. If the cutoff integral diverges, we can be sure that the entire integral (9) will also diverge.

An upper bound on Q may be estimated. The hydrodynamic results which we shall utilize require that $\omega_k\tau\ll 1$, where τ^{-1} is the collision frequency. At $T=1^\circ\text{K}$, $\tau\sim 10^{-9}$ sec and at lower temperatures $\tau\propto T^{-\alpha}$ where $\alpha>1$.^{14,15} Thus we must have $Q\ll 4\times 10^{-4}T^\alpha\text{\AA}^{-1}$. This guarantees that $\beta\omega/2\ll 1$. The value of Q becomes vanishingly small as $T\rightarrow 0$. However, we are only interested in divergences at long wavelengths and our arguments are unchanged as long as T is not strictly zero.

Hohenberg and Martin have obtained $\chi''_{\mathbf{v}_s\mathbf{v}_s}(\mathbf{k},\omega)$ exactly in the long-wavelength limit. We will not reproduce their results here, but shall describe two features of $\chi''_{\mathbf{v}_s\mathbf{v}_s}(\mathbf{k},\omega)$ which are important to us. First, $\chi''_{\mathbf{v}_s\mathbf{v}_s}(\mathbf{k},\omega)$ is very sharply peaked as a function of frequency about the values $\omega=c_1k$; c_2k . c_1 and c_2 are the velocities of first and second sound, respectively. Second, $\chi''_{\mathbf{v}_s\mathbf{v}_s}(\mathbf{k},\omega)$ approaches zero as $\omega\rightarrow 0$ at least as strongly as the first power of ω . So, since $\chi''_{\mathbf{v}_s\mathbf{v}_s}(\mathbf{k},\omega)$ contributes to (9) only in the long-wavelength limit, this function will only be nonzero for frequencies such that $\beta\omega/2\ll 1$. Then (9) becomes

$$\frac{1}{2}\langle[\varphi(\mathbf{r},t)-\varphi(\mathbf{r}',t)]^2\rangle=\frac{4\Omega_n}{\beta}\int\frac{d^n k}{(2\pi)^n}\frac{1}{k^2}\sin^2\frac{\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}{2}\times\int\frac{d\omega}{2\pi}\frac{\chi''_{\mathbf{v}_s\mathbf{v}_s}(\mathbf{k},\omega)}{\omega}. \quad (10)$$

$\chi''_{\mathbf{v}_s\mathbf{v}_s}(\mathbf{k},\omega)$ obeys a dispersion relation

$$\chi_{\mathbf{v}_s\mathbf{v}_s}(\mathbf{k},z)=\int\frac{d\omega}{\pi}\frac{\chi''_{\mathbf{v}_s\mathbf{v}_s}(\mathbf{k},\omega)}{\omega-z},$$

where z is in the upper half plane. Thus the frequency integral in (10) may be performed immediately, and

¹⁴ D. Pines, in *Many-Body Theory, 1965 Tokyo Summer Lectures in Theoretical Physics*, edited by R. Kubo (W. A. Benjamin, Inc., New York, 1966).

¹⁵ I. M. Khalatnikov, Ref. 5.

from Ref. 6 we obtain

$$\lim_{k\rightarrow 0}\int\frac{d\omega}{\pi}\frac{\chi''_{\mathbf{v}_s\mathbf{v}_s}(\mathbf{k},\omega)}{\omega}=\chi(\mathbf{k},0)=\frac{1}{\Omega_n\rho_s},$$

where ρ_s is the superfluid density. The remaining wave-number integral in (10) can be evaluated explicitly in one, two, and three dimensions. The leading terms for $Q|\mathbf{r}-\mathbf{r}'|\rightarrow\infty$ are

$$\begin{aligned} \frac{1}{2}\langle[\varphi(\mathbf{r},t)-\varphi(\mathbf{r}',t)]^2\rangle &= \frac{1}{2\beta\rho_s} \\ &\times |\mathbf{r}-\mathbf{r}'|, \quad n=1 \\ &\times \frac{1}{\pi}\ln 2Q|\mathbf{r}-\mathbf{r}'|, \quad n=2 \\ &\times \frac{1}{\pi^2}\left[Q-\frac{\pi}{2|\mathbf{r}-\mathbf{r}'|}\right], \quad n=3. \end{aligned} \quad (11)$$

Before discussing the consequences of (11), we shall investigate the higher order terms in the series of (3).

It is easily shown that all odd-order terms (i.e., $\xi=3, 5, 7, \dots$) in (3) are zero in equilibrium, since these terms are not a function of the difference variable $|\mathbf{r}-\mathbf{r}'|$. The even terms do not vanish and we cannot evaluate them explicitly. However, some qualitative arguments about their asymptotic behavior can be made. We discuss the fourth-order term and suggest that the higher order terms can be similarly treated.

The fourth-order term in the cumulant expansion is

$$\langle X^4 \rangle_c = \langle X^4 \rangle - 3\langle X^2 \rangle^2, \quad (12)$$

where $\langle X \rangle = \langle X^3 \rangle = 0$. The first term on the right-hand side of (12) has the form

$$\begin{aligned} \langle X^4 \rangle &= \langle [\varphi(\mathbf{r},t) - \varphi(\mathbf{r}',t)]^4 \rangle \\ &= \Omega_n^4 \int \frac{d^n k}{(2\pi)^n} \int \frac{d^n k'}{(2\pi)^n} \int \frac{d^n k''}{(2\pi)^n} \int \frac{d^n k'''}{(2\pi)^n} \\ &\times \langle \varphi(\mathbf{k},t) \varphi(\mathbf{k}',t) \varphi(\mathbf{k}'',t) \varphi(\mathbf{k}''',t) \rangle e^{i(\mathbf{k}+\mathbf{k}'+\mathbf{k}''+\mathbf{k}''')\cdot\mathbf{r}} \\ &\times \{ [1 - e^{-i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}][1 - e^{-i\mathbf{k}'\cdot(\mathbf{r}-\mathbf{r}')}] \\ &\times [1 - e^{-i\mathbf{k}''\cdot(\mathbf{r}-\mathbf{r}')}][1 - e^{-i\mathbf{k}'''\cdot(\mathbf{r}-\mathbf{r}')}] \}. \end{aligned} \quad (13)$$

Again, in the equilibrium system this must be a function of $|\mathbf{r}-\mathbf{r}'|$ so the correlation function in (13) must be zero unless

$$\mathbf{k}+\mathbf{k}'+\mathbf{k}''+\mathbf{k}'''=0. \quad (14)$$

There are several ways of satisfying (14). We may choose

$$\mathbf{k}''' = -(\mathbf{k}+\mathbf{k}'+\mathbf{k}''), \quad (15)$$

or the three terms which result from equating pairs of

terms to zero, i.e.,

$$\begin{aligned} \mathbf{k} &= -\mathbf{k}', & \mathbf{k}'' &= -\mathbf{k}'''; \\ \mathbf{k} &= -\mathbf{k}'', & \mathbf{k}' &= -\mathbf{k}'''; \\ \mathbf{k} &= -\mathbf{k}''', & \mathbf{k}' &= -\mathbf{k}''. \end{aligned} \quad (16)$$

The terms of (16) just lead to the factorization

$$\langle \varphi(\mathbf{k}, t) \varphi(\mathbf{k}', t) \varphi(\mathbf{k}'', t) \varphi(\mathbf{k}''', t) \rangle \rightarrow \langle \varphi(\mathbf{k}, t) \varphi(-\mathbf{k}, t) \rangle \times \langle \varphi(\mathbf{k}', t) \varphi(-\mathbf{k}', t) \rangle,$$

and since there are three such terms, they just add up to cancel the $-3\langle X^2 \rangle^2$ on the right-hand side of (12). So the only contribution to (12) consists of terms which remain when (15) is satisfied, but none of the relations (16) hold. Using (15) in (12) we are left with three integrals of the form $\int [d^n k / (2\pi)^n]$ to perform. Assuming once again that for small $|\mathbf{k}|$, $\varphi(\mathbf{k}, t) = \theta(1/k)$, we discover that

$$\langle X^4 \rangle_c = \langle [\varphi(\mathbf{r}, t) - \varphi(\mathbf{r}', t)]^4 \rangle_c \sim \left\{ \int_{k \leq Q} \frac{d^n k}{(2\pi)^n} \right\}^3 \frac{1}{k^4}. \quad (17)$$

Each of the integrals in (17) is again cut off at some small wave number Q . This then enables us to estimate the long-wavelength contributions to (17) as

$$\begin{aligned} \langle X^4 \rangle_c &\propto (-1) \times |\mathbf{r} - \mathbf{r}'|, & n=1 \\ &\times |\mathbf{r} - \mathbf{r}'|^{-2}, & n=2 \\ &\times |\mathbf{r} - \mathbf{r}'|^{-5}, & n=3. \end{aligned} \quad (18)$$

Higher order terms in the cumulant expansion would, by the same argument, continue to yield terms like $|\mathbf{r} - \mathbf{r}'|$ in one dimension, but increasingly smaller terms (higher inverse powers of $|\mathbf{r} - \mathbf{r}'|$) in two and three dimensions.

Thus, in one dimension, as $|\mathbf{r} - \mathbf{r}'| \rightarrow \infty$

$$G(\mathbf{r} - \mathbf{r}'; t) \xrightarrow{|\mathbf{r} - \mathbf{r}'| \rightarrow \infty} -in_0 \exp \left\{ -\frac{\gamma}{\beta \rho_s} |\mathbf{r} - \mathbf{r}'| \right\}, \quad (19)$$

where γ is a function of the thermodynamic parameters of the superfluid system. Equation (19) implies that the single-particle Green's function becomes vanishingly small and that there is no evidence of long-range order in the one-dimensional system.

In two dimensions the leading term in the cumulant expansion is $\ln 2Q|\mathbf{r} - \mathbf{r}'|$, the higher order terms approach zero, so

$$G(\mathbf{r} - \mathbf{r}'; t) \xrightarrow{|\mathbf{r} - \mathbf{r}'| \rightarrow \infty} -in_0 \times \exp \left\{ -\frac{1}{2\pi\beta\rho_s} \ln 2Q|\mathbf{r} - \mathbf{r}'| + \theta \left(\frac{1}{|\mathbf{r} - \mathbf{r}'|^2} \right) \right\}. \quad (20)$$

Since the leading term is computed exactly, we strongly

suggest that no long-range order will be present in two dimensions at nonzero temperatures. This means that either the phase transition will not exist for $n=2$, or if it does exist it will have a rather different character than in three dimensions. [Note added in proof. Recently P. C. Hohenberg (to be published) has confirmed these arguments, using quite general methods.]

In three dimensions the results are different. Here

$$G(\mathbf{r} - \mathbf{r}'; t) \xrightarrow{|\mathbf{r} - \mathbf{r}'| \rightarrow \infty} -in_0 \times \exp \left\{ -\frac{Q}{2\pi^2\beta\rho_s} + \theta \left(\frac{1}{|\mathbf{r} - \mathbf{r}'|} \right) \right\}. \quad (21)$$

The Green's function approaches a finite constant, indicating the presence of long-range order in the system. Notice that $\beta\rho_s \rightarrow 0$ as $T \rightarrow T_\lambda$; the off-diagonal long-range order¹ disappears at T_λ .

III. CONCLUSIONS

The analysis presented here agrees with the general theorem¹⁶ that any one-dimensional system, containing finite-range forces, will not undergo a phase transition. Further our results in three dimensions are in agreement with what we know about the bulk helium system. In two dimensions our results indicate that there is no long-range order in G .¹⁷ This implies that if the phase transition exists in two dimensions, it will have a different character than in three dimensions.¹⁸ For example, if an ordered state appeared in two dimensions, its hydrodynamic behavior would not be described by the two-fluid equations in their usual form.^{4,5}

Our results do of course depend on the assumption that the higher order superfluid velocity correlation functions are well behaved in the long-wavelength limit. (One can imagine a very delicate cancellation occurring, in say two dimensions, which results in the leading term in the expansion being proportional to $k^{-2+\eta}$ where $\eta > 0$.) We cannot justify our assumptions about the higher order correlation functions. Knowledge of these terms would require solution of the nonlinear two-fluid equations.

ACKNOWLEDGMENT

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¹⁶ L. D. Landau and E. M. Lifshitz, *Statistical Physics* (Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 1958).

¹⁷ Similar conclusions have been reached by G. V. Chester and L. Reatto (to be published).

¹⁸ F. Dyson (private communication) has argued for the existence of a phase transition in the analogous problem of a two-dimensional Heisenberg ferromagnet.