Simple Formula for Quasi-Elastic Electron Scattering*

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(Received 27 June 1966)

An approximate analytic formula is developed for high-energy (several BeV or higher) quasi-elastic electron scattering from a complex nucleus, on the basis of a simple nuclear model. The approximation should be good enough to be useful in estimating experimental counting rates and backgrounds. For scattering at an angle θ, the quasi-elastic peak is centered, in this approximation, at the energy $\epsilon_0=\epsilon/[1+2(\epsilon/M)\times\sin^2(\frac{1}{2}\theta)]$, where ϵ is the incident energy and M is the nucleon mass. The peak has a full width at half maximum of about $\sqrt{2}(\epsilon_0{}^2/\epsilon)v_+[1+(1+\epsilon/M)^2\tan^2(\frac{1}{2}\theta)]^{1/2}\sin\theta$, where v_+ is a characteristic nucleon velocity in the nucleus, $v_+\approx 0.28$. Inelastic electron scattering from complex nuclei is briefly discussed.

I. INTRODUCTION

THE main features of electron scattering from complex nuclei are well known. For a fixed incident energy and scattering angle there is a spectrum of final electron energies. Lying highest in energy is the elastic peak, followed by a number of peaks corresponding to excitation of discrete nuclear energy levels, which are in turn followed by a continuum of energies corresponding to ejection of a nucleon from the nucleus (quasi-elastic scattering) and particle production. Finally, the whole spectrum is somewhat blurred by radiative corrections.

With present experimental techniques and incident electron energies on the order of several BeV or more, experimental energy resolutions are rarely better than about 1%, giving an uncertainty in the final electron energy of tens of MeV. Thus the various discrete nuclear energy levels are no longer resolved, and experimentally there appears to be just a single elastic peak. Furthermore, the form factors of a complex nucleus[1] are such that even for small angles the elastic scattering from a complex nucleus is quite small relative to that from a single proton. At energies of several BeV, the dominant feature of the final electron-energy spectrum is therefore the quasi-elastic peak. In what follows we will be concerned only with quasi-elastic and inelastic scattering; the initial electron energy is assumed to be at least several BeV and consequently the electron is highly relativistic. We assume that the nucleus and incident electron are unpolarized, that only the scattered electron is detected, and that its polarization is not measured. Unless otherwise stated, all quantities are evaluated in the laboratory frame.

Quasi-elastic scattering is discussed in Sec. II; by making some approximations, an analytic expression is derived for the quasi-elastic peak. In Sec. III particle production (inelastic scattering) is briefly discussed, and concluding remarks are made in Sec. IV.

* Research supported by the U. S. Atomic Energy Commission, contract No. AT(30-1)2752.

[1] J. H. Fregeau, Phys. Rev. 104, 225 (1956).

II. QUASI-ELASTIC SCATTERING

In order to discuss quasi-elastic scattering, some nuclear model must be adopted. The gross features of quasi-elastic scattering are rather model-independent,[2] so we adopt the simple if somewhat unphysical Fermi-gas model. More precisely, the nucleons are assumed to populate with uniform density a sphere in momentum space of radius P_f.

A. Notation and Kinematics

It is convenient to use polar coordinates, where the initial electron momentum (**k**) defines the polar axis. The final electron momentum (**k**′) defines the angles $\Phi=0$, and θ. The electron energies ϵ and ϵ' are equal to $|\mathbf{k}|$ and $|\mathbf{k}'|$, respectively. The scattering is assumed to take place on a single nucleon moving initially with a speed v in the ϑ, φ direction, where in general $\vartheta\neq\theta$, $\varphi\neq\Phi$. There is reflection symmetry in the $\Phi=0$ plane, so we may require $0\leq\varphi\leq\pi$. Under these conditions the final electron energy is fixed by kinematics[3]:

$$\epsilon'=\frac{\epsilon(1-v\cos\vartheta)}{1+2(\epsilon/\gamma M)\sin^2(\frac{1}{2}\theta)-v\cos\alpha},\tag{1}$$

where M denotes the nucleon mass, $\gamma=(1-v^2)^{-1/2}$, and α is defined as the angle between the initial nucleon velocity and the final electron momentum

$$\cos\alpha=\cos\theta\cos\vartheta+\sin\theta\sin\vartheta\cos\varphi.$$

Because of the Fermi motion of the nucleons there is not a single well-defined energy for an electron scattered at an angle θ. Experimentally, the cross section observed is $\partial^2\sigma(\epsilon,\theta,\epsilon')/\partial\epsilon'\partial\Omega$; this is easily related to the Rosenbluth cross section for the individual nucleons

[2] J. Goldemberg and R. H. Pratt, Rev. Mod. Phys. 38, 311 (1966). This article also gives references to more refined treatments of quasi-elastic scattering; see in particular W. Czyz, Phys. Rev. 131, 2141 (1963).

[3] We use $\hbar=c=1$, and a metric such that $x_\mu=(\mathbf{x},x_4)=(\mathbf{x},it)$; $x^2=x_\mu x_\mu=\mathbf{x}^2-t^2$.

$d\sigma_R(\epsilon,\theta)/d\Omega$ (see Appendix):

$$\frac{\partial^2\sigma}{\partial\epsilon'\partial\Omega}=\sum_{\text{all nucleons}}\frac{3}{2\pi\eta_+^3}\int\eta^2 d\eta\int d(\cos\vartheta)\times gr|d\epsilon'/d\varphi|^{-1}\frac{d\tilde{\Omega}}{d\Omega}\frac{d\sigma_R(\tilde{\epsilon},\tilde{\theta})}{d\tilde{\Omega}},\quad(2)$$

with $\eta^2=\gamma^2-1$ and η_+ the maximum η of the nucleons in the nucleus. A tilde means that the quantity so labelled is to be evaluated in the Lorentz frame in which the nucleon is at rest. The Pauli exclusion principle is contained in g, defined to be unity unless the *final* nucleon momentum lies inside the Fermi sphere, in which case $g\equiv 0$. The quantity r accounts for the fact that the incident electron flux is in general not the same for an individual nucleon as it is for the nucleus as a whole[4]; $r=1-v\cos\vartheta$. For completeness we quote the Rosenbluth formula[5]:

$$\frac{d\sigma_R(\epsilon,\theta)}{d\Omega}=\frac{\alpha^2\cos^2(\frac{1}{2}\theta)}{4\epsilon^2\sin^4(\frac{1}{2}\theta)}[1+2(\epsilon/M)\sin^2(\tfrac{1}{2}\theta)]^{-1}\times\left\{\frac{G_E^2(t)}{1-t/4M^2}-(t/4M^2)\times\left[\frac{1}{1-t/4M^2}+2\tan^2(\tfrac{1}{2}\theta)\right]G_M^2(t)\right\},\quad(3)$$

where $t=-(k'-k)^2=-4\epsilon\epsilon_0\sin^2(\frac{1}{2}\theta)$, and G_E and G_M are the nuclear form factors.[6]

Equation (2) may easily be integrated numerically, and it is known that even this very simple model gives qualitatively correct results for quasi-elastic scattering.[2]

B. Analytic Formula for Quasi-Elastic Scattering

By making the approximation that $v\ll 1$, we may, within the framework of our simple model, easily integrate Eq. (2), obtaining a very simple analytic formula for quasi-elastic scattering. No approximations in incident energy or scattering angle are made within the region for which the formula is valid. For the moment we assume that $g=1$.

In the approximation that $v\ll 1$, it is clear that we may write

$$gr\frac{d\tilde{\Omega}}{d\Omega}\frac{d\sigma_R(\tilde{\epsilon},\tilde{\theta})}{d\tilde{\Omega}}\approx\frac{d\sigma_R(\epsilon,\theta)}{d\Omega},\quad(4)$$

and take it out of the Eq. (2) integral. We then are left with an essentially kinematic integral

$$\frac{3}{2\pi v_+^3}\int_{v-}^{v+}v^2dv\int_{(\cos\vartheta)-}^{(\cos\vartheta)+}d(\cos\vartheta)|d\epsilon'/d\varphi|^{-1}\equiv I(\epsilon,\theta,\epsilon').\quad(5)$$

The limits of integration are determined from Eq. (1), which may be rewritten in the small v approximation as

$$\epsilon'=\epsilon_0[1+v(\epsilon_0/\epsilon)\cos\alpha-v\cos\vartheta],\quad(1a)$$

where $\epsilon_0\equiv\epsilon[1+2(\epsilon/M)\sin^2(\frac{1}{2}\theta)]^{-1}$. In other words, ϵ_0 is just the energy that the final electron would have if the nucleon were initially at rest. From Eq. (1a) $|d\epsilon'/d\varphi|=(\epsilon_0^2/\epsilon)v\sin\theta\sin\vartheta\sin\varphi$, and Eq. (5) becomes

$$I(\epsilon,\theta,\epsilon')=\frac{3\epsilon}{2\pi v_+^3\epsilon_0^2\sin\theta}\int_{v-}^{v+}vdv\times\int_{(\cos\vartheta)-}^{(\cos\vartheta)+}d(\cos\theta)\frac{1}{\sin\vartheta\sin\varphi}.\quad(6)$$

Using Eq. (1a) again to eliminate φ, the inner integral of Eq. (6) takes the form

$$I_x=\int_{x-}^{x+}dx(ax^2+bx+c)^{-1/2},\quad(7)$$

where

$$x_\pm=(-1/a)[\tfrac{1}{2}b\pm(c-a-1)^{1/2}],$$
$$a=-1-(1+\epsilon/M)^2\tan^2(\tfrac{1}{2}\theta),$$
$$b=-2(1+\epsilon/M)[\tan(\tfrac{1}{2}\theta)][(\epsilon'-\epsilon_0)/(\epsilon_0 v\sin\theta)](\epsilon/\epsilon_0),$$
$$c=1-(\epsilon'-\epsilon_0)^2(\epsilon/\epsilon_0)/(\epsilon_0 v\sin\theta)^2.$$

On evaluating the Eq. (7) integral, one finds

$$I_x=\pi[1+(1+\epsilon/M)^2\tan^2(\tfrac{1}{2}\theta)]^{-1/2}.\quad(8)$$

This result is perhaps surprising. I_x is independent of both v and ϵ', as long as ϵ' is a kinematically possible energy for the given v, In other words, if a nucleus consisted of nucleons all moving at the same speed ($v\ll 1$), with effective spherical symmetry in momentum space, then the quasi-elastic energy spectrum would be white (constant) between two sharp limits. This very simple result makes the remaining integration in Eq. (6) trivial, and we finally have for the quasi-elastic cross section

$$\frac{\partial^2\sigma}{\partial\epsilon'\partial\Omega}=\sum_{\text{all nucleons}}\frac{d\sigma_R(\epsilon,\theta)}{d\Omega}\frac{3}{4\Delta\epsilon'}\left[1-\left(\frac{\epsilon'-\epsilon_0}{\Delta\epsilon'}\right)^2\right],\quad(9)$$

where

$$\Delta\epsilon'=(\epsilon_0^2/\epsilon)v_+[1+(1+\epsilon/M)^2\tan^2(\tfrac{1}{2}\theta)]^{1/2}\sin\theta.$$

This cross section holds (within the approximations made) for $\epsilon_0-\Delta\epsilon'<\epsilon'<\epsilon_0+\Delta\epsilon'$, and is zero elsewhere.

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[4] See, e.g., J. M. Jauch and F. Rohrlich, *The Theory of Photons and Electrons* (Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 1955), p. 166.

[5] As given in G. Källén, *Elementary Particle Physics* (Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 1964), p. 225.

[6] L. N. Hand, D. G. Miller, and Richard Wilson, Rev. Mod. Phys. **35**, 335 (1963), Eqs. (47), (48), (51), and (52); see also L. H. Chan, K. W. Chen, J. R. Dunning, Jr., N. F. Ramsey, J. K. Walker, and Richard Wilson, Phys. Rev. **141**, 1298 (1966).

In Fig. 1 a comparison is shown between the approximate Eq. (9) and a Monte Carlo calculation based on Eq. (2). The absence of "wings" is of course the fault of the overly simple nuclear model, rather than of the approximation $v \ll 1$. The deviation of Eqs. (2) and (9) from each other gives an indication of the error induced by the approximation $v \ll 1$. Although Eq. (9) is seen to give a rather crude approximation to the cross section, it should suffice in estimating counting rates and backgrounds in electron-scattering experiments. Restating Eq. (9) in words, the quasi-elastic peak should be centered at a final energy corresponding to scattering from a free, stationary nucleon. The peak should be roughly parabolic in shape, with a full width at half maximum of $\sqrt{2}\Delta\epsilon'$. Two special cases may be of interest: If $(1+\epsilon/M)^2 \tan^2(\frac{1}{2}\theta) \ll 1$, as is often the case in current experiments, $\Delta\epsilon \approx \epsilon_0 v_+ \sin\theta$. At energies and angles such that $(1+\epsilon/M)\tan(\frac{1}{2}\theta) \gg 1$, and $(2\epsilon/M)\sin^2(\frac{1}{2}\theta) \gg 1$, $\Delta\epsilon' \approx v_+\epsilon_0$, $\epsilon_0 \approx M/[2\sin^2(\frac{1}{2}\theta)]$. A reasonable value for v_+ is on the order of 0.28, and is more or less independent of the nucleus in question.

It is tempting to apply Eq. (8) in dealing with a more realistic nucleon momentum distribution, since the remaining integral would be easy to do explicitly. This would give a more reasonable cross section for ϵ' far from ϵ_0, but in just this region the $v \ll 1$ approximation breaks down. In the small-v region (ϵ' near ϵ_0), on the other hand, any reasonable momentum distribution is roughly like that of a Fermi gas.

In the discussion leading to Eq. (9), g was set equal to unity; we have not considered the effect of the Fermi exclusion principle as it applies to the recoiling nucleon. In order for scattering to occur, the momentum of the recoiling nucleon must lie outside the Fermi sphere. If the three-momentum transfer to the nucleon is $\Delta k = |\mathbf{k}' - \mathbf{k}|$, no nucleon is affected by the exclusion principle if $\Delta k > 2P_f$. For $\Delta k < 2P_f$ the quasi-elastic scattering is suppressed, that at high ϵ' being affected most, as shown in Fig. 2. The integrated quasi-elastic

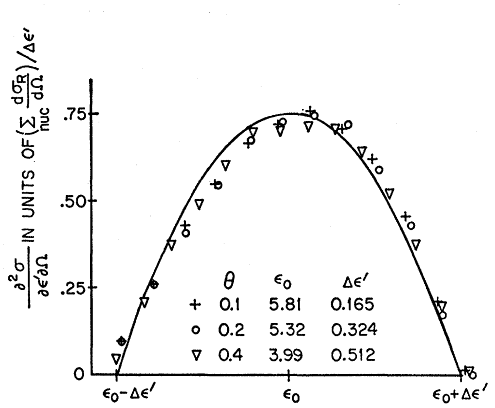

FIG. 1. The quasi-elastic peak at 6 BeV for various angles. The smooth curve is the Eq. (9) approximation. The energies are given in BeV, the angle in radians.

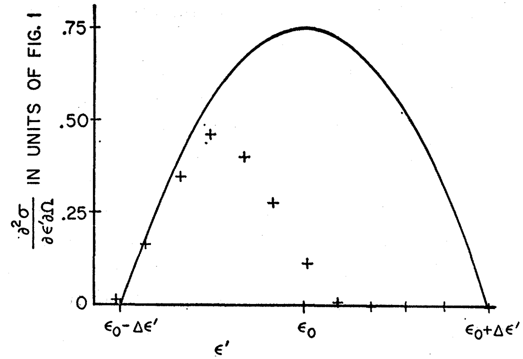

FIG. 2. The quasi-elastic peak at 6 BeV, $\theta=0.025$ rad. The smooth curve is the Eq. (9) approximation, which fails badly, since $\Delta k < 2P_f$. $\epsilon_0 = 5.988$ BeV, $\Delta\epsilon' = 43$ MeV.

cross section, on this model, is given by the well-known formula

$$\int \frac{\partial^2\sigma(\epsilon,\theta,\epsilon')}{\partial\epsilon'\partial\Omega} d\epsilon' = \tfrac{3}{4}(\Delta k/P_f)[1-\tfrac{1}{12}(\Delta k/P_f)^2]\frac{d\sigma_R(\epsilon,\theta)}{d\Omega}, \quad (10)$$

for $\Delta k \le 2P_f$. Since a good value for P_f is on the order of 270 MeV, this effect may not be very important for incident energies of several BeV or greater at experimentally feasible scattering angles. In Fig. 2 a comparison is made between the approximation (Eq. (9)] and the exact formula [Eq. (2)], with $\theta \approx 1\frac{1}{2}°$ at 6 BeV. At this very small angle the three-momentum transfer is about 150 MeV, so the exclusion principle is important.

III. INELASTIC SCATTERING

Just as the quasi-elastic scattering has been discussed on the Fermi-gas nuclear model, the inelastic scattering may be estimated. In this case we consider an electron incident on a (moving) nucleon, producing a final state (nucleon and pion, for example) of invariant mass M'. Equation (1) becomes in this case

$$\epsilon' = \frac{\epsilon(1-v\cos\vartheta) - (M'^2 - M^2)/2\gamma M}{1+2(\epsilon/\gamma M)\sin^2(\frac{1}{2}\theta) - v\cos\alpha}. \quad (1b)$$

Equation (2) is equally easily generalized to

$$\frac{\partial^2\sigma}{\partial\epsilon'\partial\Omega} = \sum_{\text{all nucleons}} \frac{3}{2\pi\eta_+^3}\int d\epsilon'' \int \eta^2 d\eta \times \int d(\cos\vartheta) g r |d\epsilon'/d\varphi|^{-1} \times \frac{\partial(\tilde{\epsilon}'',\tilde{\Omega})}{\partial(\epsilon'',\Omega)} \frac{\partial^2\sigma_{ep}(\tilde{\epsilon},\tilde{\theta}',\tilde{\epsilon}')}{\partial\tilde{\epsilon}''\partial\tilde{\Omega}}, \quad (11)$$

where σ_{ep} is the electroproduction cross section on a single nucleon target. In Eq. (11), g is not so easily dealt with as before, since the recoiling nucleon momentum is kinematically undermined. Also, if most of the single-pion production goes by N^* production, $g=1$ regardless of the final nucleon momentum. (The N^* decay, however, might be inhibited by the exclusion principle until the N^* left the nucleus.) Apart from this question regarding g, Eq. (11) should give a rough approximation to the entire inelastic scattering cross section from a complex nucleus, provided the electroproduction cross section is known over the necessary kinematic range.

IV. CONCLUSIONS

Over a wide range of energies and angles, Eq. (9) should yield a fair approximation ($\approx 20\%$) to the central region of the quasi-elastic peak. The regions far from the center ($\epsilon'=\epsilon_0$) of the peak will be badly represented for two reasons: in these regions the nuclear model is too crude, and one may not say that $v\ll 1$. One could of course give Eq. (9) a more realistic appearance by interpreting the square bracket as an expansion of $\exp[(\epsilon'-\epsilon_0)^2/\Delta\epsilon'^2]$, but the resulting equation would also be unreliable for ϵ' far from ϵ_0. In order for Eq. (9) to be valid, the three-momentum transfer to the nucleon must be greater than about 550 MeV. For smaller momentum transfers the peak will become more narrow, primarily at the expense of the $\epsilon'>\epsilon_0$ side. The area under the peak will be reduced according to Eq. (10). If the nucleon form factors were to vary rapidly with q^2 in the region of interest (if, for example, there were a diffraction zero), Eq. (9) would be a poor approximation; thus far, however, the form factors appear quite smooth, without zeros. For some energies and angles the quasi-elastic peak will overlap the inelastic region, which one may test using Eq. (1b). It is possible that the above analysis will give better agreement with experiment by using an "effective nucleon mass" smaller than M. Thus far initial energies have been too low for this to be an easily observable effect; nuclear energy levels have complicated the picture. Finally, discussing in detail the radiative corrections to this crude model would be gilding the dandelion, but the qualitative effect will, of course, be to suppress the $\epsilon'>\epsilon_0$ region and enhance the $\epsilon'<\epsilon_0$ region, making the peak somewhat asymmetric.

ACKNOWLEDGMENTS

We would like to thank Professor F. M. Pipkin and W. Faissler for interesting discussions.

APPENDIX

To relate the quasi-elastic scattering from a complex nucleus to the Rosenbluth formula for single nucleons, we consider an element of the nucleon momentum space, of volume $\delta V=v^2\delta v\delta(\cos\vartheta)\delta\varphi$. The nucleons in this element of momentum space yield scattered electrons in an energy range $\delta\epsilon'$. The cross section for these nucleons is therefore

$$\frac{\partial^2\sigma'}{\partial\epsilon'\partial\Omega}=\frac{1}{\delta\epsilon'}\frac{d\tilde{\Omega}}{d\Omega}\frac{d\sigma_R}{d\tilde{\Omega}}, \tag{A1}$$

where a tilde indicates that the quantity so labelled is to be evaluated in the rest frame of the nucleon. It is easy to show that

$$d\tilde{\Omega}/d\Omega=(1-v\cos\alpha)^{-1}[1+v(\cos\alpha-v)/(1-v\cos\alpha)]. \tag{A2}$$

It is now tempting to write for the quasi-elastic cross section due to a single nucleon

$$\frac{\partial^2\sigma}{\partial\epsilon'\partial\Omega}=\frac{1}{V}\sum_{\text{Vol}}\frac{\partial^2\sigma'}{\partial\epsilon'\partial\Omega}\delta V. \tag{A3}$$

For any given ϵ', of course, Eq. (1) fixes one of the quantities v, $\cos\vartheta$, or φ, given the other two, so the sum is really carried out over a two-dimensional surface. Equation (A3) is not quite complete, however, since the interaction rate due to the δV volume is proportional to the product of $\partial^2\sigma'/\partial\epsilon'\partial\Omega$ and flux. If the flux were the same throughout V, Eq. (A3) would be correct. In Eq. (A1), however, we have implicitly assumed that the total cross section is Lorentz-invariant, which in turn implies that the flux is proportional to $[(P\cdot k)^2-P^2k^2]^{1/2}$, where P and k are the initial four-momenta of the nucleon and electron, respectively.[4] The ratio r of the δV flux to the flux incident on the nucleus is easily found to be $r=1-v\cos\vartheta$. Equation (A3) now becomes [passing to an integral and using Eq. (1) to determine φ]

$$\frac{\partial^2\sigma}{\partial\epsilon'\partial\Omega}=\frac{1}{V}\int\eta^2d\eta\int d(\cos\vartheta)r\left|\frac{d\epsilon'}{d\varphi}\right|^{-1}\frac{d\tilde{\Omega}}{d\Omega}\frac{d\sigma_R}{d\tilde{\Omega}}. \tag{A4}$$

Equation (A4), with the inclusion of the effects of the Pauli exclusion principle, (the factor g, defined in the text) yields Eq. (2).