Nature of Singularities in Relativistic World Models

JAROSLAV PACHNER
Astronomical Institute of Czechoslovak Academy of Sciences, Prague, Czechoslovakia
(Received 27 December 1965; revised manuscript received 23 February 1966.)

It is shown that the puzzle of a discontinuous inversion of velocities of particles at the moment of maximum contraction of Friedman universes is removed by the assumption that the square root of the reciprocal absolute value of the Riemannian curvature of the three-dimensional space takes on not only positive, but also negative values. Further it is proved that the replacement of the usual uniform and isotropic distribution of the cosmic dust by a spherically symmetric one prevents contraction of the world model to a zero volume, but it does not prevent the occurrence of singularities in the density of matter at the center of symmetry. However, it may be expected that the abandonment of all symmetry in the distribution of matter, as we observe in the actual universe, in connection with the fact that no sudden inversion of velocities takes place, will also avoid the occurrence of these singularities.

INTRODUCTION

As has been recently shown by Hoyle,[1] the observational data seem to testify in favor of the opinion that our universe is oscillating. However, in the finite Friedman model there exists the well-known singularity at which the density of matter and the time derivative of the curvature radius of space become infinite at the moment when all matter is contracted into a single point. The aim of the present paper is to show that the occurrence of this singularity is an unavoidable and, at the same time, quite natural consequence of the usual assumption of physical properties and distribution of the material contents of the world model, and that the Einstein field equations, without any modification[2] and without introducing a high negative pressure,[3] give an oscillating world model whose volume never contracts to zero, as soon as the assumption of a uniform and isotopic distribution of matter is abandoned.

NATURE OF SINGULARITIES IN NEWTONIAN COSMOLOGY AND IN FRIEDMAN UNIVERSES

From the elementary kinetic theory of gases it is well known that an ideal gas fills a zero volume at zero absolute temperature, because its molecules are represented by dimensionless mass points. Analogously the particles of the cosmic dust with a vanishing pressure are to be considered as dimensionless mass points too, in consequence of which the volume of the universe with all its material contents can contract without any difficulty down to a mathematical point.

In the framework of Newtonian cosmology it is scarcely possible to imagine a physical process by means of which the matter contracting to a point with the highest velocity can be suddenly stopped and sent back expanding with the inverse velocity. If we exclude an exactly central collision of particles, we are forced to suppose that the dimensionless mass points, having reached the singular point in the process of contraction, cross it and continue their paths in the same direction. This situation of regular oscillations about a point is mathematically described by the continuity of the first time derivative of the scale factor, which may also take negative values.

In the metric of the Friedman universes,

$$ds^2=-F^2\{(1-kr^2/R_0^2)^{-1}dr^2+r^2d\Omega^2\}+dt^2, \quad (1.1)$$

with

$$d\Omega^2=d\vartheta^2+\sin^2\vartheta d\varphi^2, \quad (1.2)$$

the Riemannian curvature K_R of the three dimensional space is spatially constant and given by the expression[4]

$$K_R=k/F^2R_0^2. \quad (1.3)$$

According to the value of $k=0, \pm1$, the space is either flat, or positively or negatively curved. The positive square root of the reciprocal absolute value of the Riemannian curvature $|F/k|R_0$ is called the radius of curvature of the three-dimensional space. Since the spatial volume V of the whole universe (if $k=+1$), or of its part (for $k=0, -1$), is determined by the formula

$$V=2\int_0^{R_0}\int_0^{\pi}\int_0^{2\pi}|\det g_{ij}|^{1/2}drd\vartheta d\varphi\approx|F|^3, \quad (i, j=r, \vartheta, \varphi), \quad (1.4)$$

the mass density ρ is inversely proportional to the third power of the absolute value of F. The field equation for the (4,4) component, which suffices for determining the function F, takes, after a few reductions, the form

$$(dF/d\chi)^2+k=|F|^{-1}, \quad (1.5)$$

where

$$\chi=t/R_0. \quad (1.6)$$

The mass density ρ is expressed by the relation

$$\rho=(3/8\pi R_0^2|F|^3). \quad (1.7)$$

[1] F. Hoyle, Nature **208**, 111 (1965).
[2] F. Hoyle and J. V. Narlikar, Proc. Roy. Soc. (London) **A278**, 465 (1964). Cf. also: S. Deser and F. A. E. Pirani, *ibid.* **A288**, 133 (1965).
[3] J. Pachner, Monthly Notices Roy. Astron. Soc. **131**, 173 (1965); Bull. Astron. Inst. Czech. **16**, 321 (1965).
[4] See, for instance, L. D. Landau and E. M. Lifshitz, *Teorija polja* (Moscow, 1962), 4th edition, p. 380ff.

Hence we may compute the positive constant R_0 considering the mass density $\rho=\rho_1$ as known at the moment $t=t_1$ at which $F=F_1$. All the quantities are measured in the geometrical system of units in which the velocity of light and the Newtonian constant of gravitation are dimensionless numbers equal to unity.

Usually the function $F(\chi)$ is assumed to be always positive and even, and its first derivative discontinuous at the moment $\chi=0$ (for $k=0, -1$), or at the moments $\chi=n\pi$, $n=0, \pm1, \pm2, \cdots$ (for $k=+1$). However, none of the preceding relations will be changed, if we suppose that the function $F(\chi)$ may take negative as well as positive values. The integral of Eq. (1.5) is then represented in the flat space ($k=0$) by the function

$$F=(\tfrac{3}{2})^{2/3}\chi^{1/3}|\chi|^{1/3}. \quad (1.8)$$

In the positively curved space ($k=+1$) we express it in the parametric form

$$\chi=\xi-\cos\xi\,\sin\xi, \quad (1.9a)$$

$$F=(1-\cos^2\xi)^{1/2}\sin\xi. \quad (1.9b)$$

In the negatively curved space ($k=-1$) we have

$$\chi=\cosh\xi\,\sinh\xi-\xi, \quad (1.10a)$$

$$F=(\cosh^2\xi-1)^{1/2}\sinh\xi. \quad (1.10b)$$

This function $F(\chi)$, defined by (1.8), (1.9a), (1.9b), (1.10a), and (1.10b), respectively, is odd and no discontinuities appear in its first derivative.

The Hubble law does not change its form:

$$dx_i/dt=[(dF/d\chi)/F](x_i/R_0) \quad (1.11)$$

either, but each of Cartesian coordinates x_i, denoting the position of the mass point, takes alternately positive and negative values.

We have thus shown that a mere replacement of F by its absolute value $|F|$ on the right-hand side of the field Eq. (1.5), which follows from the expression (1.4) and from the definition of the mass density and is compatible with the relations of the Riemannian geometry, solves the puzzle of a sudden inversion of velocities of the mass points at the moment of maximum contraction of Friedman universes.

BEHAVIOR OF A RELATIVISTIC WORLD MODEL WITH SPHERICALLY SYMMETRIC DISTRIBUTION OF MATTER

In this section we shall prove that the volume of a relativistic oscillating world model in which the assumption of a uniform and isotropic distribution of matter is replaced by a spherically symmetric one, never contracts periodically to zero.

For this purpose let us consider the exact spherically symmetric solutions of Einstein field equations found by Datt[5] and classified recently by the author.[6] Without any loss of generality we may, in order to simplify our deductions, put into the most general form of the metric,[6] written in a co-moving system of coordinates, the arbitrary function $\sigma(r)$ equal to $r^2\psi(r)$. In the positively curved space we have

$$ds^2=-F^2\{(1-r^2\psi)^{-1}K^2dr^2+r^2d\Omega^2\}+dt^2, \quad (2.1)$$

where (with $s=0$ for expansion, and $s=1$ for contraction)

$$K=1+(-1)^s r[\pm t(d\psi^{1/2}/dr) + (d\eta/dr)]F^{-1}(|F|^{-1}-1)^{1/2}, \quad (2.2)$$

$$0\leqq r\leqq R_0,\quad 0\leqq\vartheta\leqq\pi,\quad 0\leqq\varphi\leqq 2\pi. \quad (2.3)$$

The arbitrary real function $\psi(r)$ is restricted by the condition $0\leqq r^2\psi\leqq 1$. The signs of equality hold here for both boundary values of r. The function η is a further arbitrary real function of r. The function $F(\chi)$, defined by the differential equation (1.5) with $k=+1$, is described by Eqs. (1.9), but the variable χ is here given by the relation

$$\chi=\pm t\psi^{1/2}+\eta. \quad (2.4)$$

The density of matter is determined by the formula[6a]

$$8\pi\rho=(1/K|F|^3)r^{-2}(d/dr)(r^3\psi). \quad (2.5)$$

The spatial volume of the world model is to be computed by the formula (1.4) whereinto we put the components of the tensor $g_{\mu\nu}$ from the metric (2.1). Since the integrand in (1.4) is positive, the volume can become zero only if at a certain moment the function $F(\chi)$ vanishes simultaneously for all the admissible values of r.

Let us suppose that at the moment $t=t_0$ all matter is concentrated into a single singular point. If this state with zero volume of space had to appear at the moment $t=t_0+T_0$ again, the conditions

$$\psi^{1/2}=\pi/T_0=\text{const},\quad \eta=-\pi(t_0/T_0)=\text{const}$$

in Eq. (2.4) (with the plus sign) should be satisfied.

Let us now suppose that at the moment $t=0$ there exists a given distribution of matter. All matter will contract after a certain time interval, say T, into a single point, if the function $\eta(r)$ is so chosen as to satisfy the equation [following from (2.4) with the plus sign]

$$\pi=T\psi^{1/2}+\eta.$$

However, this singular state will repeatedly appear with the period T_0 only if

$$T_0\psi^{1/2}=\pi.$$

The necessary and sufficient conditions for the periodical occurrence of the singular state in which all matter of the world model is concentrated into a single

[5] B. Datt, Z. Physik 108, 314 (1938).

[6] J. Pachner, Bull. Astron. Inst. Czech. 17, 105 (1966).

[6a] Note added in proof. The density becomes singular either at the center of symmetry ($F=0$), or at a moment when two spherical surfaces with the radii r and $(r+dr)$ reach the same physical position ($K=0$). In the region where $K<0$ the formula (2.5) must not be applied, because the same physical point is here occupied by two points with different values of the coordinate r and with different densities.

point are therefore expressed by the relations

$$\psi^{1/2}=\eta/t_0=1/R_0=\text{const}, \quad t_0=\text{const}, \qquad (2.6)$$

i.e., the variable χ must not depend on the radial coordinate r. Putting the conditions (2.6) into the preceding equations, we obtain

$$\chi=(t+t_0)/R_0, \quad K=1. \qquad (2.7)$$

The metric and the density are then given by Eqs. (1.1) and (1.7), respectively. We find that the consequence of the conditions (2.6) is the metric of the oscillating Friedman universe with a uniform and isotropic distribution of matter.

This result can be easily understood in terms of Newtonian cosmology. All matter, having exploded from a single point at the moment $t=t_0$, can fall back in it at the moment $t=t_0+T_0$ only if the velocity is distributed strictly according to the Hubble law (1.11) and if, in addition, the matter is strictly uniformly and isotropically distributed in order to avoid perturbations in velocity distribution caused by gravitational interaction. In the other case with a given initial distribution of matter we can achieve a simultaneous contraction of all matter into a single singular point by a suitable choice of velocity distribution (taking into account, of course, gravitational interaction too), but this singular state never reappears, if the distribution of matter is not strictly uniform and isotropic from the very outset.

CONCLUDING REMARKS

In the preceding section we have found that already the displacement of the assumption of a uniform and isotropic distribution of the cosmic dust by a spherically symmetric one prevents the contraction of the world model to a zero volume, for the particular concentric shells reach the center of symmetry $r=0$ at different moments. In spite of the nonzero volume of the world model, the density of matter becomes at these moments singular at the center of symmetry, as follows from Eq. (2.5) and also agrees with Hawking.[7] However, it may be expected that the abandonment of all symmetry in the distribution of matter, as we observe in the actual universe, will prevent the occurrence of singularities in the density of matter too, if we remember that instead of a sudden inversion of velocities of the particles at the center of symmetry the strong interaction in the state of maximum contraction of the universe will certainly cause their scattering. It is perhaps possible from the observational data of the clusters of galaxies[8] to draw some conclusions on the size of the volume and the behavior of the universe in its most contracted stage and to find here the origin of the rotational velocities of galaxies.

It seems therefore that Bonnor[9] was quite right when he expressed his conviction that the fact that certain quantities in our differential equations became infinite at the start of the cosmic expansion was nothing more than an indication of the breakdown of the theory. In this paper it has been shown that the Einstein theory itself is not wrong, but the simplifying assumption of a uniform and isotropic distribution of matter (which may be used in the present epoch of the cosmic evolution) must not be applied to the description of the states of the highest contraction of the universe where it gives results not corresponding to the physical reality.

In conclusion let us remark that we encounter an analogous situation also in the case of a gravitational collapse.[10] The assumption of a uniform and isotropic distribution of matter inside the collapsing body made by Oppenheimer and Snyder[11] and by Faulkner, Hoyle, and Narlikar[12] must not be used, because the metric is then not continuous across the boundary of the body. A non-uniform, spherically symmetric distribution of the material dust with a vanishing pressure prevents its simultaneous contraction into a singular point,[10] but it does not prevent, as Penrose has also shown,[13] the occurrence of singularities in the density of matter at the center of the body.

[7] S. W. Hawking, Phys. Rev. Letters **15**, 689 (1965).

[8] F. Zwicky and K. Rudnicki, Astrophys. J. **137**, 707 (1963); F. Zwicky and J. Berger, *ibid.* **141**, 34 (1965); F. Zwicky and M. Karpowicz, *ibid.* **142**, 625 (1965).

[9] H. Bondi, W. B. Bonnor, R. A. Lyttleton, and G. J. Wittrow, *Rival Theories in Cosmology* (Oxford University Press, London, 1960), pp. 4–6.

[10] J. Pachner, Bull. Astron. Inst. Czech. **17**, 108 (1966).

[11] J. R. Oppenheimer and H. Snyder, Phys. Rev. **56**, 459 (1939).

[12] J. Faulkner, F. Hoyle, and J. V. Narlikar, Astrophys. J. **140**, 1100 (1964).

[13] R. Penrose, Phys. Rev. Letters **14**, 57 (1965).