

Einstein, Infeld, and Hoffmann Problem of Motion in Five Space

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The method of Einstein, Infeld, and Hoffmann is generalized to five space. It is shown that the equations of motion that emerge in the Newtonian approximation are *partial differential* equations and that certain entities appearing may be interpreted as mass and charge. In addition to the equations of motion there appear two additional linear partial differential equations for each particle involving a third parameter as well as the mass and charge. Even for an isolated "particle" or systems of "particles" far removed from one another *partial differential* equations of motion persist. Application to an electron indicates that the three coordinates satisfy an equation in the space characterized by the additional dimension and time which implies the propagation of a wave with the velocity of the order of 10^{21} times the velocity of light in this space.

1. INTRODUCTION

MANY models for the description of the gravitational and electromagnetic field exist. It seems that the most successful formalism developed is still that of Kaluza,¹ with considerable and notable improvements by Bergmann² since it reproduces exactly the classic equations of the electromagnetic field with gravity. In spite of this success there still remain the well known difficulties associated with the classic treatment. This has led to generalizations which lead to substantially different equations and were made in the hope that there could be included within the new frameworks an adequate theory encompassing quantum-phenomena as well. It appears that these hopes have not been realized. However, it is most difficult to relinquish a theory which in the large has been subject to confirmation—quite strikingly in recent times using techniques based upon the Mössbauer effect.³ In addition to numerous verifications of the original version of the theory of relativity, Einstein, Infeld, and Hoffmann⁴ have shown by means of an ingenious approximation method that the motion of matter, represented as point singularities of the field, is sufficiently determined by the gravitational equations for empty space. The nonlinear character of the field equations and the differential identities satisfied by them are directly responsible for the interaction terms present in the equations of motion. Thus, it is not necessary to supplement the field equations with ponderomotive equations.

It is of some interest to investigate the problem of motion in a five dimensional setting to study departures from the usual results. Our immediate experiences have been most successfully described in a three dimensional space using concepts such as volume, surface, and line integrals with considerable physical significance imputed

to these quantities. The important roles played by ordinary surface integrals in the E.I.H. problem are well known. The aforementioned theories^{1,2} and subsequent generalizations^{2,5} have used various devices to rid the final equations of the coordinate of the additional dimension, it being argued that as far as we are able to ascertain at present it is sufficient to use the ordinary 3 plus 1 space in macro-like situations. Can we then infer that we can bridge the chasm that exists between the macro and the micro domains of physics by relaxing the dimensionality requirement? If we do, then since no *immediate* significance may be attached to symbols involving the additional "nonphysical" coordinates, an unphysical situation would ensue and thus render calculations made with them apparently meaningless. We will undertake to see, nevertheless, what happens if we strip the previous simple five dimensional theories of their adornments and maintain only the relationship involving the vanishing of the contracted curvature tensor in five space in an application to a most fundamental and primitive problem; namely, the well-known problem of motion. If we adhere to the notion that a "particle" is subject to adequate geometrical description by the specification of its three coordinates together with the requirement that the interaction between "particles" *resembles* that in the usual case, then it will be seen that in the five dimensional case the "trajectory" of the "particle" is characterized by a two dimensional rather than a one dimensional manifold. This would imply in view of a presumed inability to observe one of the parameters describing the "trajectory" that the "particle" is localized to be within a surface which latter would indicate that the paths of the "particles," insofar as one's ability to predict them, possess a diffuseness. This latter, then, may be considered as possibly characterizing the significance to be attributed to the additional parameter in such a theory.

The simplest generalization in this direction would be effected by merely taking the usual field equations in five space

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 0, \quad \mu, \nu = 0, 1, 2, 3, 4, \quad (1)$$

¹ T. Kaluza, Sitzber. preuss. Akad. Wiss., Physik.-math. Kl. **12/13**, 966 (1921).

² P. G. Bergmann, *An Introduction to the Theory of Relativity* (Prentice-Hall, Inc., New York, 1946), p. 254.

³ R. V. Pound and G. A. Rebka, Jr., Phys. Rev. Letters **3**, 439 (1959). H. J. Hay, J. P. Schiffer, T. E. Cranshaw, and P. A. Egelstaff, Phys. Rev. Letters **4**, 165 (1960). C. W. Sherwin, Phys. Rev. **120**, 17 (1960).

⁴ A. Einstein, L. Infeld, and B. Hoffmann, Ann. Math. **39**, 65 (1938). A. Einstein and L. Infeld, Ann. Math. **41**, 455 (1940). (Hereafter referred to as E.I.H.)

⁵ W. Pauli, *Theory of Relativity* (Pergamon Press, New York, 1958), p. 227.

where $g_{\mu\nu}$ is the metric tensor, $R_{\mu\nu}$ the Riemann-Christoffel curvature tensor and R its contraction. The additional dimension will be assumed to have a space-like character. "0" indices will be considered temporal, the others spatial with the "4" index reserved for the additional dimension.

2. APPROXIMATION METHOD

To obtain approximate solutions of (1) corresponding as closely as possible to the four dimensional case it is only necessary to assume additionally that differentiation with respect to, x_4 , leads to quantities of the *same* order of smallness as differentiation with respect to $x_0 (=ct)$.⁴ Upon defining

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \quad (2)$$

$$\gamma_{\mu\nu} = h_{\mu\nu} - (\eta_{\mu\nu}/2)\eta^{\alpha\beta}h_{\alpha\beta}, \quad (3)$$

(1) when multiplied by two may be written as

$$\Phi_{\mu\nu} + 2\Lambda_{\mu\nu} = 0, \quad (4)$$

where

$$\begin{aligned} \Phi_{00} &\equiv -\gamma_{00|ss} + \gamma_{st|st}, \\ \Phi_{0s} &\equiv -(\gamma_{0s|t} - \gamma_{0t|s})|t, \\ \Phi_{mn} &\equiv -(\gamma_{mn|s} - \gamma_{ms|n} - \delta_{ms}\gamma_{nt|t} + \delta_{mn}\gamma_{st|t})|s, \\ \Phi_{44} &\equiv -\gamma_{44|ss} - \gamma_{st|st}, \\ \Phi_{04} &\equiv -\gamma_{04|ss}, \\ \Phi_{4s} &\equiv -(\gamma_{4s|t} - \gamma_{4t|s})|t; \\ 2\Lambda_{00} &= -\gamma_{00|44} + \gamma_{44|44} + 2\gamma_{4t|4t} + 2\Lambda_{00}', \\ 2\Lambda_{0s} &= \gamma_{st|t0} - \gamma_{00|s0} - \gamma_{0s|44} + \gamma_{04|s4} + \gamma_{4s|04} + 2\Lambda_{0s}', \\ 2\Lambda_{mn} &= \gamma_{mn|00} - \gamma_{m0|n0} - \gamma_{n0|m0} + 2\delta_{mn}\gamma_{0t|0t} - \delta_{mn}\gamma_{00|00} \\ &\quad - \gamma_{mn|44} + \gamma_{m4|n4} + \gamma_{n4|m4} - 2\delta_{mn}\gamma_{4t|4t} \\ &\quad + 2\delta_{mn}\gamma_{04|04} - \delta_{mn}\gamma_{44|44} + 2\Lambda_{mn}', \\ 2\Lambda_{44} &= 2\gamma_{0t|0t} - \gamma_{00|00} + \gamma_{44|00} + 2\Lambda_{44}', \\ 2\Lambda_{04} &= \gamma_{0s|4s} - \gamma_{00|40} + \gamma_{4s|0s} + \gamma_{44|04} + 2\Lambda_{04}', \\ 2\Lambda_{4s} &= \gamma_{st|4t} + \gamma_{4s|00} + \gamma_{44|s4} - \gamma_{40|s0} - \gamma_{s0|40} + 2\Lambda_{4s}'; \end{aligned} \quad (5)$$

$\eta_{\mu\nu}$ denotes the Galilean metric tensor appropriately generalized ($\eta_{44} = -1$); $(\)_{|\alpha}$ denotes ordinary partial differentiation; $\Lambda_{\mu\nu}$ depicts the nonlinear contributions which are readily computed in terms of the deviations $h_{\mu\nu}$ from the Galilean metric $\eta_{\mu\nu}$; and all latin indices run from 1 to 3.

Examination of (5) discloses the existence of an additional Φ ; namely, Φ_{4s} in (5) which may be considered to be the s th component of the curl of a vector as well as Φ_{0s} and say Φ_{ts} . The introduction now of nontensor coordinate conditions which we find it convenient to take as

$$\gamma_{\mu 0|0} - \gamma_{\mu s|s} - \gamma_{\mu 4|4} = 0, \quad \mu = 0, 1, 2, 3, 4; \quad s = 1, 2, 3, \quad (7)$$

in view of the existence of five differential identities

$$(R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R)_{;\nu} = 0, \quad (8)$$

where $(\)_{;\nu}$ denotes covariant differentiation, enables one to essentially reduce the problem in its broadest outlines to the E.I.H. procedure⁴ as we see further upon examining the structure of (4), (5), and (6) and recalling that we reckon differentiation with respect to, x_4 , as leading to expressions of the same order of smallness as differentiation with respect to, $x_0 (=ct)$.

As in E.I.H. we consider the γ 's expanded as

$$\begin{aligned} \gamma_{00} &= \lambda^2 \gamma_{00} + \lambda^4 \gamma_{00} + \dots, \\ \gamma_{0s} &= \lambda^3 \gamma_{0s} + \lambda^5 \gamma_{0s} + \dots, \\ \gamma_{40} &= \lambda^2 \gamma_{40} + \lambda^4 \gamma_{40} + \dots, \\ \gamma_{44} &= \lambda^2 \gamma_{44} + \lambda^4 \gamma_{44} + \dots, \\ \gamma_{4s} &= \lambda^3 \gamma_{4s} + \lambda^5 \gamma_{4s} + \dots, \\ \gamma_{st} &= \lambda^4 \gamma_{st} + \lambda^6 \gamma_{st} + \dots, \\ s, t &= 1, 2, 3, \end{aligned} \quad (9)$$

and note that the additional dimension leads to the appearance of the expressions involving γ_{40} , γ_{44} and γ_{4s} in (9). In the lowest order of approximation (4) yields in view of the structure of $\Lambda_{\mu\nu}$

$$\begin{aligned} 2\gamma_{00,ss} &= 0, \\ 2\gamma_{44,ss} &= 0, \\ 2\gamma_{40,ss} &= 0, \end{aligned} \quad (10)$$

where $(\)_{,s}$ denotes partial differentiation with respect to x^s and the usual summation convention is used. The solutions of (10) representing "point particles" as in the E.I.H. case are taken to be

$$\begin{aligned} 2\gamma_{00} &= -\sum_j 2M_j V_j, \\ 2\gamma_{44} &= -\sum_j 2P_j V_j, \\ 2\gamma_{40} &= -\sum_j 2Q_j V_j, \end{aligned} \quad (11)$$

where

$$V_j \equiv [(x^s - \xi_j^s)(x^s - \xi_j^s)]^{-\frac{1}{2}}, \quad (12)$$

ξ_j^s , the s th coordinate of the j th "particle"; M_j and Q_j are constants proportional to the "mass" and "charge" of the j th particle, respectively. P_j may be considered another parameter describing another property of the j th particle. However, unlike the four dimensional case M_j (as well as Q_j and P_j in our case) need not be constant. Arguments given by Bergmann² indicating the constancy of M_j for the four dimensional case when generalized here yield instead the following partial differential equations for M_j , Q_j and P_j

$$\begin{aligned} -M_{j,0} + Q_{j,4} &= 0, \\ Q_{j,0} - P_{j,4} &= 0, \end{aligned} \quad (13)$$

in the lowest order of approximation. (13) follows upon noting that the surface integrals of G_{0s} and G_{4s} about the j th singularity vanish for $j=1, 2, \dots$. Now the introduction of (11) and (13) into our "coordinate con-

dition" equations (7) yield

$$\begin{aligned} 3\gamma_{0s} &= \sum_j (2M_j \xi_{j,0^s} - 2Q_j \xi_{j,4^s}) V_j, \\ 3\gamma_{4s} &= \sum_j (2Q_j \xi_{j,0^s} - 2P_j \xi_{j,4^s}) V_j, \end{aligned} \quad (14)$$

and we note that $3\gamma_{0s}$ and $3\gamma_{4s}$ are also simple three dimensional harmonic functions.

Now the Newtonian approximation follows in the usual way upon insisting that the surface integral of $4\Lambda_{st}$ vanish. A lengthy calculation yields

$$\begin{aligned} 2\Lambda_{st} &= 1/36(2\gamma_{00,s}\gamma_{00,t} - 20\gamma_{00,s}\gamma_{44,t} \\ &\quad - 20\gamma_{00,t}\gamma_{44,s} + 2\gamma_{44,s}\gamma_{44,t} - 24\gamma_{00}\gamma_{00,st} \\ &\quad - 12\gamma_{00}\gamma_{44,st} - 12\gamma_{44}\gamma_{00,st} - 24\gamma_{44}\gamma_{44,st} \\ &\quad + 11\delta_{st}\gamma_{00,r}\gamma_{00,r} + 32\delta_{st}\gamma_{00,r}\gamma_{44,r} \\ &\quad + 11\delta_{st}\gamma_{44,r}\gamma_{44,r}) - \frac{3}{2}\delta_{st}\gamma_{40,r}\gamma_{40,r} \\ &\quad + 2\gamma_{40,s}\gamma_{40,t} + 2\gamma_{40}\gamma_{40,st} + 4\Psi_{st}, \end{aligned} \quad (15)$$

where $4\Psi_{st}$ is comprised of terms other than $2\Lambda_{st}'$ and is obtained from the third equation of the right hand side of (6) (linear in the γ 's). If we denote by $\gamma_{00}^{(i)}$, $\gamma_{40}^{(i)}$, and $\gamma_{44}^{(i)}$ the corresponding γ_{00} , γ_{40} , and γ_{44} given by (11) with the V_j term omitted and x^s replaced by $\xi_j^{(s)}$, then the surface integral calculations yield

$$\begin{aligned} 2M_j \xi_{j,00^s} - 4Q_j \xi_{j,40^s} + 2P_j \xi_{j,44^s} \\ + \frac{1}{9}(13M_j - 4P_j)\partial\gamma_{00}^{(j)}/\partial\xi_j^s \\ + \frac{1}{9}(13P_j - 4M_j)\partial\gamma_{44}^{(j)}/\partial\xi_j^s \\ - 2Q_j\partial\gamma_{40}^{(j)}/\partial\xi_j^s = 0, \end{aligned} \quad (16)$$

in the Newtonian approximation.

3. SUMMARY OF LOWEST ORDER CALCULATIONS, NEWTONIAN APPROXIMATION

In order that (16) reduce to the customary equations in the limit $P_j=0$, it is only necessary to put

$$\begin{aligned} M_j &= 9\kappa m_j/13, \\ Q_j &= 3(\kappa/26)^{1/2} q_j, \\ P_j &= (9\kappa/13)p_j, \end{aligned} \quad (17)$$

where κ is the gravitational constant, m_j the mass in grams and q_j the charge in e.s.u. The parameter, P_j , is adjusted for convenience to involve the parameter, p_j , proportional to it. With these units and the physically significant entities brought in focus, (16) may be written as the following *partial differential* equations with, t , and, x_4 , playing the role of independent variables

$$\begin{aligned} \partial^2 \xi_j^s / \partial t^2 - \frac{1}{3}(cq_j/m_j)(26/\kappa)^{1/2} \partial^2 \xi_j^s / \partial x^4 \partial t \\ + (p_j c^2/m_j) \partial^2 \xi_j^s / (\partial x^4)^2 \\ - \sum_{i \neq j} [\kappa m_i + (q_i q_j)/m_j - 4\kappa p_i m_i / (13m_j) \\ + \kappa p_i p_j / m_j - 4\kappa p_i / 13] \partial V(i,j) / \partial \xi_j^s = 0, \end{aligned} \quad (18)$$

where

$$V(i,j) = [(\xi_i^s - \xi_j^s)(\xi_i^s - \xi_j^s)]^{-1/2}; \quad (19)$$

with, t , in seconds and, x^4 , the additional coordinate expressed in centimeter units. (13) on the other hand takes the form

$$\begin{aligned} \partial m_j / \partial t - (13/18\kappa)^{1/2} c (\partial q_j / \partial x^4) &= 0, \\ \partial q_j / \partial t - (18\kappa/13)^{1/2} c (\partial p_j / \partial x^4) &= 0. \end{aligned} \quad (20)$$

4. THE "FREE PARTICLE"

Equation (18) indicates that even for an isolated "particle" or systems of "particles" far removed from one another the coordinates ξ_j^s ($s=1, 2, 3$) of the j th particle satisfy the second order equation obtained by considering only the first three terms of (18) which we express as

$$(\partial/\partial t - c_{j+}\partial/\partial x^4)(\partial/\partial t - c_{j-}\partial/\partial x^4)\xi_j^s = 0, \quad (21)$$

in the event m_j , q_j and p_j are presumed constant, an assumption quite consistent with (20). $c_{j\pm}$ is given by

$$\begin{aligned} c_{j\pm} &= \{(q_j/m_j)(18\kappa/13)^{-1/2} \\ &\quad \pm [(q_j/m_j)^2(18\kappa/13)^{-1} - p_j/m_j]^{1/2}\}c. \end{aligned} \quad (22)$$

(21) implies with the f_j^s and g_j^s arbitrary functions of the indicated arguments that

$$\xi_j^s = f_j^s(c_{j+}t + x^4) + g_j^s(c_{j-}t + x^4), \quad (23)$$

a wave motion for the, ξ_j^s , which is propagated with velocities $-c_{j\pm}$ in our $t-x^4$ space.

5. CONCLUSION

Equations (18) and (20) indicate that if ξ_j^s is independent of x^4 and p_j is nil, then (18) reduces to the ordinary Newtonian equations with the Coulomb interaction term included. In general, however, unlike the usual case where the trajectory of the "particles" are curves in three space, the "particles" satisfying (18) and (20) may be considered distributed on an ordinary surface in three space. This is due to the circumstance that each of the three ξ_j^s involves the two independent variables t and x^4 so that elimination of the latter would establish an equation involving the ξ_j^1 , ξ_j^2 , and ξ_j^3 . It is possible to consider in the same way a further generalization to any arbitrary number of dimensions each time considering, insofar as the approximation method is concerned, differentiation with respect to the new dimensions to have the same magnitude properties as that cited for x^4 in comparison with x_0 differentiation.

Furthermore, the insistence that (21) be a hyperbolic partial differential equation has as a consequence $p_j/m_j < 0$ to assure this property even when $q_j=0$. For an electron, the velocity of propagation attains a value of the order 10^{21} times the velocity of light! One would be forced to conclude that if it were possible to communicate in such a space, it would be perhaps a simple task to assess the state of the distant reaches of the

universe within a considerably shorter interval of time than heretofore and so indeed to consider occurrences almost anywhere to be of a contemporary nature. In any event the structure of the equations of motion which

have emerged even for "free particles" may perhaps provide us with other conceptual possibilities the starting point of which may lead to a theory with less startling consequences.

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Derivation of the *CPT* Theorem and the Connection between Spin and Statistics from Postulates of the *S*-Matrix Theory*

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The *CPT* theorem and the normal connection between spin and statistics are shown to be consequences of postulates of the *S*-matrix approach to elementary particle physics. The postulates are much weaker than those of field theory. Neither local fields nor any reference to space-time points are used. Quantum commutation relations and properties of the vacuum play no role. Completeness of the asymptotic states and positive definiteness of the metric are not required, though certain weaker asymptotic conditions prevail. The proofs depend on unitarity, macroscopic relativistic invariance, and a very weak analyticity requirement on the mass-shell scattering functions. The proofs are in the framework of the new *S*-matrix approach to elementary particle physics, which is established on a formal basis.

I. INTRODUCTION

THE two most important general physical consequences of relativistic field theory are the *CPT* theorem¹⁻⁴ and the connection between spin and statistics.⁵⁻¹⁰ The *CPT* theorem states that for every process occurring in nature there is an allowed dual process in which the particles of the first are replaced by their respective antiparticles, all spins are reversed, and paths are changed to their images under inversion through the origin in space-time. Relationships between probabilities are stated to be the same for a process and its dual. The proved connection between spin and statistics is that wave functions are symmetric under the interchange of variables referring to two identical integral-spin particles and antisymmetric for the half-integral-spin case.

These important results are derived from the postulates of local field theory, which, however, are subject to considerable doubt. In the first place it is not known whether the postulates are sufficiently realistic to

include any theories except trivial ones in which the scattering matrix is unity.

Secondly, the postulates are very specialized and restrictive, in that they assign a fundamental role to hypothetical local field operators defined over the field of space-time points. Experience does not entail the existence of such points, and the restriction to theories in which they play a fundamental role may immediately exclude all theories connected to physical reality. Because space-time points are experimentally inaccessible both in practice and in principle, their introduction runs counter to the philosophy of quantum mechanics. This philosophic inconsistency appears to have its analog in the mathematical structure in which related inconsistencies seem to arise.¹¹⁻¹³

Even within the general framework of local field theories, some of the postulates are so restrictive that many reasonable theories are excluded. In particular, the requirements of the completeness of the asymptotic states and the positive definiteness of the metric are assumed to hold, not only asymptotically, but also throughout the course of the interaction. But added states of negative metric not among those observed asymptotically seem to be exactly what are needed to remove the apparent inconsistencies from field theory. A theory based on this possibility is among those being

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