Electromagnetic Properties of Superconductors*

K. K. GUPTA† AND V. S. MATHUR
The Enrico Fermi Institute for Nuclear Studies and the Department of Physics, The University of Chicago, Chicago, Illinois
(Received February 2, 1959)

An integral relationship between the super-current and the vector potential of the electromagnetic field, similar to Pippard's phenomenological equation, is derived from the gauge-invariant theory of Wentzel. Expressions for coherence distance and penetration depth have been calculated.

1. INTRODUCTION

It is known that the derivation of the electromagnetic properties of superconductors in the theory of Bardeen, Cooper, and Schrieffer[1] suffers from lack of gauge invariance.[2] An explicitly gauge-invariant derivation of the Meissner effect has recently been given by Wentzel,[3] using Bogoliubov's mathematical approach[4] to the BCS model of superconductivity. In this note we obtain, starting from Wentzel's gauge-invariant expression for the supercurrent, an integral or nonlocal relationship between the current $\mathbf{j}(\mathbf{r})$ and the vector potential $\mathbf{A}(\mathbf{r})$, as first proposed by Pippard.[5] Our derivation neglects all temperature and Coulomb effects, as has been done in Wentzel's paper. We find that our coherence distance differs by a factor $1/\pi$ from the BCS value. We further obtain the Fourier transform of the integral equation. The penetration depth in the long-wavelength limit (London limit) is found to differ by a factor $(2/\rho)^{\frac{1}{2}}$ from the BCS value, as already shown by Wentzel, ρ being the Bogoliubov constant. In the short-wavelength limit (Pippard limit), it differs by a factor $(2/\pi\rho)^{\frac{1}{3}}$ from their value.

Recently a derivation of the Meissner effect and Pippard equation has also been attempted by Rickayzen[6] using Bogoliubov's method. However, his derivation is not gauge-invariant, as has been pointed out by Wentzel.[3]

2. DERIVATION OF PIPPARD EQUATION

Wentzel starts with the gauge-invariant Hamiltonian

$$H=H_0+H_g+H_A+H_{AA}, \tag{1}$$

where H_0 is the energy of the free electron and phonon gas, H_g the electron-phonon interaction with coupling constant g, and H_A+H_{AA} the interaction of the electrons with the electromagnetic field described by the vector potential $\mathbf{A}$. In order to be able to apply Bogoliubov's method in a gauge-invariant way, he makes a canonical transformation which brings (1) to the form

$$H'=e^{-L}He^{L},$$

where L satisfies the equation

$$[H_0+H_g,\, L]=-H_A.$$

Since L is linear in A, we obtain, neglecting terms of order A^2,

$$H'=H_0+H_g,$$

which is the Hamiltonian in the absence of the electromagnetic field. The current density,

$$\mathbf{J}=-(1/V)\partial H/\partial\mathbf{A}=\mathbf{j}_0+\mathbf{j}_A,$$

is correspondingly transformed into

$$\begin{aligned}\mathbf{j}'&=e^{-L}\mathbf{j}e^{L}\\&=\mathbf{j}_0+\mathbf{j}_A+[\mathbf{j}_0,L]+O(A^2).\end{aligned}$$

Expanding L in powers of g, one can write

$$\mathbf{j}'=\mathbf{j}_0+\mathbf{j}_A'+\mathbf{j}_{gA}+\mathbf{j}_{ggA}+\cdots.$$

Wentzel now shows that certain "exchange terms" of the expectation value of $\mathbf{j}_{ggA}$ in the unperturbed Bogoliubov ground state, which are absent in the ordinary perturbation theory, give rise to a supercurrent. These terms, in the special gauge $\operatorname{div}\mathbf{A}=0$, are

$$\begin{aligned}\mathbf{j}(\mathbf{r})=&\frac{4e^2}{m}\frac{g^2}{(2\pi)^9}\int d\mathbf{r}'d\mathbf{q}d\mathbf{l}d\mathbf{l}'(uv)_l(uv)_{l'}e^{i\mathbf{q}\cdot(\mathbf{r}-\mathbf{r}')}\\
&\times\Bigg\{\frac{\mathbf{l}\cdot\mathbf{A}(\mathbf{r}')}{\mathbf{q}\cdot(\mathbf{l}+\frac{1}{2}\mathbf{q})}\frac{1}{2E(\mathbf{l}')-E(\mathbf{l}+\mathbf{q})-E(\mathbf{l})}\\
&\times\Bigg[(\mathbf{l}+\mathbf{q})\frac{\omega^2(\mathbf{l}'+\mathbf{l}+\mathbf{q})}{\omega^2(\mathbf{l}'+\mathbf{l}+\mathbf{q})-[E(\mathbf{l}')-E(\mathbf{l})]^2}\\
&+\mathbf{l}\frac{\omega^2(\mathbf{l}'+\mathbf{l})}{\omega^2(\mathbf{l}'+\mathbf{l})-[E(\mathbf{l}+\mathbf{q})-E(\mathbf{l}')]^2}\Bigg]\\
&-\frac{\mathbf{l}'\cdot\mathbf{A}(\mathbf{r}')}{\mathbf{q}\cdot(\mathbf{l}'+\frac{1}{2}\mathbf{q})}\frac{1}{2E(\mathbf{l}')-E(\mathbf{l}+\mathbf{q})-E(\mathbf{l})}\\
&\times\Bigg[(\mathbf{l}+\mathbf{q})\frac{\omega^2(\mathbf{l}'+\mathbf{l}+\mathbf{q})}{\omega^2(\mathbf{l}'+\mathbf{l}+\mathbf{q})-[E(\mathbf{l}')-E(\mathbf{l})]^2}\\
&+\mathbf{l}\frac{\omega^2(\mathbf{l}-\mathbf{l}')}{\omega^2(\mathbf{l}-\mathbf{l}')-[E(\mathbf{l}+\mathbf{q})-E(\mathbf{l}')]^2}\Bigg]\Bigg\}.\end{aligned} \tag{2}$$

---

* Work done under the auspices of the U. S. Atomic Energy Commission.

† On leave of absence from the Tata Institute for Fundamental Research, Bombay, India.

[1] Bardeen, Cooper, and Schrieffer, Phys. Rev. **108**, 1175 (1957), hereafter referred to as BCS.

[2] M. J. Buckingham, Nuovo cimento **5**, 1763 (1957).

[3] G. Wentzel, Phys. Rev. **111**, 1488 (1958).

[4] N. N. Bogoliubov, Nuovo cimento **7**, 794 (1958).

[5] A. B. Pippard, Proc. Roy. Soc. (London) **A216**, 547 (1953).

[6] G. Rickayzen, Phys. Rev. **111**, 817 (1958).

Here $\mathbf{l}$, $\mathbf{l}'$ are the momenta of electrons, $\mathbf{q}$ the momentum transfer with the electromagnetic field, $\omega(\mathbf{l})$ the energy of a phonon of momentum $\mathbf{l}$, $E(\mathbf{l})$ that of an electron of momentum $\mathbf{l}$, and

$$(uv)_l=\tfrac{1}{2}c\{c^2+[E(\mathbf{l})-E_F]^2\}^{-\frac{1}{2}}, \tag{3}$$

c being the energy gap, and E_F the energy of an electron at the surface of the Fermi sphere. It is understood that at singularities one takes the principal values of the integrals. We use the natural units in which $\hbar=1$, velocity of light$=1$. We shall replace factors like $\omega^2/\{\omega^2-[E(\mathbf{l}')-E(\mathbf{l})]^2\}$ by unity, since the main contributions to the integrals arise from the neighborhood of the Fermi surface, where $[E(\mathbf{l}')-E(\mathbf{l})]^2\lesssim c^2 \ll\bar{\omega}^2$, $\bar{\omega}$ being the mean phonon energy.

The second term in the curly brackets in Eq. (2) can easily be shown to be zero. In the remaining term we put $\mathbf{l}+\mathbf{q}=\mathbf{k}$, and perform integrations over the directions of $\mathbf{l}$ and $\mathbf{k}$ making use of the following[1]

$$\int_{-1}^{+1} d(\cos\theta)\ \mathbf{k}e^{\pm i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}=\frac{2}{\pm i}\frac{f(k,R)}{kR^2}\nabla R,$$

where

$$R=|\mathbf{r}-\mathbf{r}'|,$$
$$f(k,R)=kR\cos kR-\sin kR.$$

We then obtain

$$\mathbf{j}(\mathbf{r})=\frac{2}{\pi^6}\frac{e^2g^2}{m}\int d\mathbf{r}'\,\frac{\nabla R[\mathbf{A}(\mathbf{r}')\cdot\nabla R]}{R^4}G(R), \tag{4}$$

with

$$G(R)=\int\frac{f(l,R)f(k,R)}{k^2-l^2}\frac{1}{2E(l')-E(l)-E(k)}\times(uv)_l(uv)_{l'}ldl\,kdk\,l'^2dl'. \tag{5}$$

Now let

$$\xi=E(l)-E_F,\quad \xi'=E(l')-E_F,\quad \xi_k=E(k)-E_F,$$
$$\eta=(\xi+\xi_k)/2.$$

Since the maximum contribution to the integral comes from regions near the Fermi surface, we take[1]

$$f(l,R)f(k,R)\simeq\tfrac{1}{2}k_F{}^2R^2\cos[(\xi_k-\xi)R/v_F]\quad\text{for } R\gg k_F{}^{-1}. \tag{6}$$

Then Eq. (5) becomes

$$G(R)=-\tfrac{1}{32}c^2m^2k_F{}^3R^2I(\alpha), \tag{7}$$

$$I(\alpha)=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\frac{\cos[\alpha(\eta-\xi)]}{(\eta-\xi)(\eta-\xi')(c^2+\xi^2)^{\frac{1}{2}}(c^2+\xi'^2)^{\frac{1}{2}}}\times d\xi d\xi' d\eta, \tag{8}$$

where

$$\alpha=2R/v_F. \tag{9}$$

To perform the integrations in $I(\alpha)$, we note that

$$\frac{dI(\alpha)}{d\alpha}=-\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\frac{\sin[\alpha(\eta-\xi)]}{(\eta-\xi')(c^2+\xi^2)^{\frac{1}{2}}(c^2+\xi'^2)^{\frac{1}{2}}}\times d\xi d\xi' d\eta. \tag{10}$$

Now

$$\int_{-\infty}^{+\infty}\frac{\sin[\alpha(\eta-\xi)]}{(\eta-\xi')}d\eta=\pi\cos\alpha\xi\cos\alpha\xi',$$

so that Eq. (10) becomes

$$dI(\alpha)/d\alpha=-4\pi K_0{}^2(\alpha c), \tag{11}$$

where

$$K_0(x)=\int_0^{\infty}\frac{\cos x\xi}{(1+\xi^2)^{\frac{1}{2}}}d\xi \tag{12}$$

is the modified Bessel function of the second kind.[7]

From Eq. (8) one sees that $I(\infty)=0$. Hence

$$I(\alpha)=\frac{4\pi}{c}\int_{\alpha c}^{\infty}K_0{}^2(y)dy. \tag{13}$$

From Eqs. (13), (7), and (4), we finally obtain

$$\mathbf{j}(\mathbf{r})=-\frac{e^2g^2}{4\pi^5}mck_F{}^3\frac{\pi^2}{4}\int d\mathbf{r}'\frac{\mathbf{R}[\mathbf{A}(\mathbf{r}')\cdot\mathbf{R}]}{R^4}J(R), \tag{14}$$

where we have put

$$J(R)=\frac{4}{\pi^2}\int_{\alpha c}^{\infty}K_0{}^2(y)dy. \tag{15}$$

The factor $4/\pi^2$ in the definition of $J(R)$ is chosen to make $J(0)=1$.

Now making use of the following expressions for the Bogoliubov constant ρ and the London coefficient Λ,

$$\rho=g^2mk_F/2\pi^2,$$
$$\Lambda=m/ne^2=3\pi^2m/k_F{}^3e^2,$$

the current density can be written in the phenomenological form of Pippard:

$$\mathbf{j}(\mathbf{r})=-\frac{3}{4\pi}\frac{\rho}{2\Lambda}\frac{1}{\xi_0}\int d\mathbf{r}'\frac{\mathbf{R}[\mathbf{A}(\mathbf{r}')\cdot\mathbf{R}]}{R^4}J(R). \tag{16}$$

Here we have introduced the constant

$$\xi_0\equiv\int_0^{\infty}J(R)dR=\frac{v_F}{\pi^2c}. \tag{17}$$

Equation (16) is not identical to the original Pippard equation

$$\mathbf{j}(\mathbf{r})=-\frac{3}{4\pi}\frac{1}{\Lambda}\frac{1}{\xi_0}\int d\mathbf{r}'\frac{\mathbf{R}[\mathbf{A}(\mathbf{r}')\cdot\mathbf{R}]}{R^4}e^{-R/\xi_0}, \tag{18}$$

[7] Erdelyi, Magnus, Oberhettinger, and Tricomi, *Higher Transcendental Functions* (McGraw-Hill Book Company, Inc., New York, 1953), Vol. 2.

but differs by a factor $\rho/2$, and in the form of the kernel $J(R)$. Our kernel $J(R)$ is, however, quite similar to Pippard's as can be seen from Fig. 1. They coincide at $R=0$ and at $R=\infty$, and their integrals over R from 0 to ∞ have the same value ξ_0. Therefore ξ_0 defined by Eq. (17) can be interpreted as the coherence distance. It differs from the BCS value by a factor $1/\pi$. It may also be written in the form suggested by Faber and Pippard:

$$\xi_0=\left(\frac{kT_c}{\pi^2 c}\right)\left(\frac{v_F}{kT_c}\right)=0.06\left(\frac{v_F}{kT_c}\right),$$

where, following BCS, c/kT_c has been taken to be 1.75. The numerical factor 0.06 is to be compared with the experimental values of 0.15 by Faber and Pippard,[8] and of 0.27 by Glover and Tinkham.[9] A precise comparison with experimental data is, however, not very meaningful in view of the simplifications involved in the model.

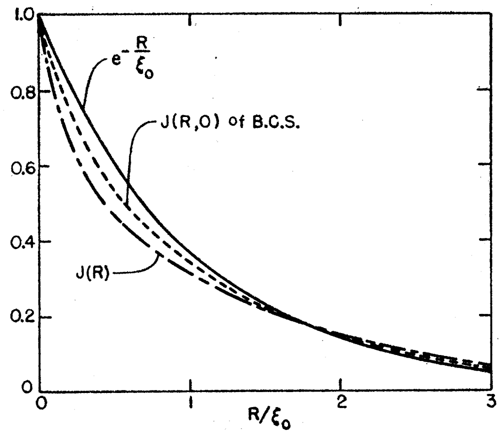

FIG. 1. The kernel $J(R)$ for the current density at $T=0°K$ compared with the Pippard kernel e^{-R/ξ_0} and the BCS kernel $J(R,0)$.

3. FOURIER TRANSFORM OF THE PIPPARD INTEGRAL

The Fourier transform of Eq. (16) may be written as

$$\mathbf{j}(\mathbf{q})=-(1/4\pi)K(q)\mathbf{A}(\mathbf{q})$$
$$=\frac{3}{4\pi}\frac{\rho}{2\Lambda}\frac{1}{\xi_0}\int\frac{\mathbf{R}[\mathbf{R}\cdot\mathbf{A}(\mathbf{q})]}{R^4}e^{-i\mathbf{q}\cdot\mathbf{R}}J(\mathbf{R})d\mathbf{R}. \quad (19)$$

The angular integrations can be performed by choosing the polar axis in the direction of $\mathbf{q}$. We then get

$$K(q)=3\pi\frac{\rho}{2\Lambda}\frac{1}{\xi_0}\int_0^\infty dR\int_{-1}^{+1}du(1-u^2)e^{-iRqu}J(R), \quad (20)$$

where $u=\cos\theta$, θ being the polar angle. Upon using Eq. (15) for $J(R)$, the R-integral in Eq. (20) can be written as

$$I=\int_0^\infty K_0^2(yR)Re^{-iRqu}dR. \quad (21)$$

Now using the following representations[7] of the modified Bessel functions,

$$K_0^2(x)=2\int_0^\infty K_0(2x\cosh t)dt,$$

$$K_0(x)=\int_0^\infty e^{-x\cosh v}dv,$$

Eq. (21) can be written as

$$I=2\int_0^\infty\int_0^\infty\frac{1}{(2y\cosh t\cosh v+iqu)^2}dtdv, \quad (22)$$

where integration over R has been performed. Substituting this in Eq. (20), and carrying out integration over y, we obtain

$$K(q)=\frac{48c}{\pi v_F}\frac{\rho}{2\Lambda}\frac{1}{\xi_0}\int_{-1}^{+1}du\int_0^\infty dt\int_0^\infty dv$$
$$\times\frac{(1-u^2)}{q^2u^2+(16c^2/v_F^2)\cosh^2 t\cosh^2 v}. \quad (23)$$

In the long-wavelength limit (London limit), $qv_F/c\to 0$, we easily find that

$$K(q)\simeq\frac{4v_F}{\pi c}\frac{\rho}{2\Lambda}\frac{1}{\xi_0},$$

whence

$$\mathbf{j}(\mathbf{q})=-\frac{v_F}{\pi^2 c}\frac{\rho}{2\Lambda}\frac{1}{\xi_0}\mathbf{A}(\mathbf{q}), \quad (24)$$

which agrees with Eq. (42) in Wentzel's paper.[3]

In order to evaluate the kernel for larger values of q (Pippard limit), $c\ll qv_F$, we rewrite Eq. (23) as

$$K(q)=\alpha\int_0^\infty dt\int_0^\infty dv\int_0^1 du(1-u^2)$$
$$\times\frac{d}{du}\left\{\frac{1}{q\beta\cosh t\cosh v}\tan^{-1}\left(\frac{uq}{\beta\cosh t\cosh v}\right)\right\}, \quad (25)$$

where we have put

$$\alpha=\frac{96c}{\pi v_F}\frac{\rho}{2\Lambda}\frac{1}{\xi_0}, \quad (26)$$

and

$$\beta=4c/v_F. \quad (27)$$

[8] T. E. Faber and A. B. Pippard, Proc. Roy. Soc. (London) **A231**, 53 (1955).
[9] R. E. Glover and M. Tinkham, Phys. Rev. **104**, 844 (1956).

Upon integrating over u by parts, Eq. (25) becomes

$$K(q)=2\alpha\int_0^\infty dt\int_0^\infty dv\int_0^1 du\frac{u}{q\beta\cosh t\cosh v}\times\tan^{-1}\left(\frac{uq}{\beta\cosh t\cosh v}\right). \quad (28)$$

Now making use of the identity

$$\tan^{-1}x=\pi/2-\tan^{-1}(1/x),$$

and of the result $\int_0^\infty dx/\cosh x=\pi/2$, we obtain

$$K(q)=\frac{\alpha}{q\beta}\left(\frac{\pi}{2}\right)^3+I(q), \quad (29)$$

where

$$I(q)=-\frac{2\alpha}{q\beta}\int_0^\infty dt\int_0^\infty dv\int_0^1 du\frac{u}{\cosh t\cosh v}\times\tan^{-1}\left(\frac{\beta\cosh t\cosh v}{uq}\right). \quad (30)$$

We shall now estimate the integral $I(q)$ and show that it contributes terms of order $(c/v_Fq)^2$. We first split the range of integration of u into two intervals, $0\leqslant u\leqslant\beta/q$ and $\beta/q\leqslant u\leqslant 1$. In the first interval, the argument of $\tan^{-1}(\beta\cosh t\cosh v/uq)$ is >1, and so we use the expansion $\tan^{-1}x=\pi/2-1/x+1/3x^3\cdots$ and integrate term by term. For the second interval, we again divide the range of v into two parts:

$$\left.\begin{array}{l}0\leqslant v\leqslant\cosh^{-1}(uq/\beta)\\ \cosh^{-1}(uq/\beta)\leqslant v\leqslant\infty\end{array}\right\}\beta/q\leqslant u\leqslant 1. \quad \begin{array}{r}(A)\\(B)\end{array}$$

In the interval (B), the argument of

$$\tan^{-1}(\beta\cosh t\cosh v/uq)$$

is again >1, and the same expansion of $\tan^{-1}$ may be used, as before. The integral (30) in range (A) is

$$-\frac{2\alpha}{q\beta}\int_{\beta/q}^1 du\int_0^{\operatorname{arc\,cosh}(uq/\beta)} dv\int_0^\infty dt\frac{u}{\cosh t\cosh v}\times\tan^{-1}\left(\frac{\beta\cosh t\cosh v}{uq}\right). \quad (31)$$

Here the integral over t is of the type

$$\int_0^\infty\frac{dt}{\cosh t}\tan^{-1}(a\cosh t)\quad\text{with } a<1,$$

which may be evaluated again by splitting the range of integration at $\cosh^{-1}(1/a)$. The result is

$$\int_0^\infty\frac{dt}{\cosh t}\tan^{-1}(a\cosh t)\simeq a\ln\left(\frac{1}{a}\right).$$

Then (31) becomes

$$-\frac{2\alpha}{q^2}\int_{\beta/q}^1 du\int_0^{\operatorname{arc\,cosh}(uq/\beta)} dv\left\{\ln\left(\frac{uq}{\beta}\right)-\ln\cosh v\right\}.$$

In the integral involving $\ln(\cosh v)$, we divide the range of integration of v at 1; in the first interval we put $\ln(\cosh v)\simeq v^2/2$, and in the second $\simeq\ln(e^v/2)$. The integral (31) then becomes $\simeq-(\alpha/q^2)[\ln(q/\beta)]^2$. This turns out to be the major contribution to $I(q)$, Eq. (30). Thus finally for $q\gg\beta$,

$$K(q)\simeq\left(\frac{\pi}{2}\right)^3\frac{\alpha}{q\beta}\left\{1-\frac{8}{\pi^3}\frac{\beta}{q}\left[\ln\left(\frac{q}{\beta}\right)\right]^2+\cdots\right\}. \quad (32)$$

The leading term of $K(q)$ may be written, by using Eqs. (26) and (27), as

$$K(q)=3\pi^2\frac{\rho}{2\Lambda}\frac{1}{q\xi_0}. \quad (33)$$

4. PENETRATION DEPTH

The kernels in the London and Pippard limits are, respectively,

$$K(q)=(\rho/2)(4\pi/\Lambda),\quad(v_Fq/c\ll 1),$$
$$K(q)=(\pi\rho/2)(3\pi^3c/\Lambda v_Fq),\quad(v_Fq/c\gg 1).$$

A comparison with the corresponding expressions of BCS shows that in the London limit our $K(q)$ is smaller by a factor $\rho/2$, and in the Pippard limit by $\pi\rho/2$. In the London limit, therefore, our penetration depth is larger by a factor $(2/\rho)^{\frac{1}{2}}$, as already shown by Wentzel. In the Pippard limit, however, the increase of penetration depth [as defined by Eq. (5.55) of BCS] is only by a factor $(2/\pi\rho)^{\frac{1}{3}}$.

ACKNOWLEDGMENTS

We are deeply indebted to Professor G. Wentzel for suggesting the problem and helpful discussions. One of us (V.S.M.) would like to thank the Government of India for the award of a scholarship.