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OSCILLATING SYSTEMS DAMPED BY RESISTANCE PROPORTIONAL TO THE SQUARE OF THE VELOCITY.

By J. Parker Van Zandt.

I. Introduction.

All bodies moving through a resisting medium encounter forces which affect their motion. In order to predetermine the motion of any dynamical system it is necessary to know the factors influencing the resisting forces and the laws governing their variations. The resistance may influence the motion very markedly; thus in the motion of large bodies at relatively high speeds, as in the propulsion of aëroplanes or of ships, the amount and variation of the resistance are dominant factors and determine the maximum velocity obtainable. A comprehensive study of the laws of damping is difficult because of the many variable factors influencing the forces of resistance. Since in many of the more important problems of engineering the predominant factor is velocity, we shall restrict the discussion to the variation of the forces of resistance with the velocity of the moving body. The resistance is considered as varying only with integral powers of the velocity, although experimentally it has been shown that the exponent of the resistance term is often fractional. Such a restriction is necessary because it is extremely difficult if not impossible to treat mathematically the irrational expressions which would otherwise occur. From a practical standpoint, the introduction of irrational or fractional exponents is in most cases an unnecessary refinement because of the presence of other indeterminate factors, partially independent of the speed, which modify the damping.

It is sometimes assumed that the majority of forms of resisted motion fall into the class in which the opposing forces are directly proportional to the velocity. Many important engineering problems, however, involve resisting forces which are proportional to the square of the velocity. Thus in hydrodynamics the resistance to the motion of water in pipes, conduits and surge towers, and the resistance to the motion of vessels, are approximately proportional to the square of the speed. When the resisting medium is air we have such problems as those of ballistics, of aërodynamics, of resistance to the motion of trains, and of air damping in some forms of electrical meters. One of the first to recognize the presence of the "square law" of resistance in the motion of projectiles through air was Sir Isaac Newton.[1] He observed the time of fall of spheres dropped from St. Paul's Cathedral and verified the law for moderate speeds. It has been found that for speeds approaching the velocity of sound, the effect of the elasticity of the fluid is felt and the index may rise considerably above its value when the resistance is a pure quadratic function. Comparing the results of German, Dutch, Russian and English experimenters, Cranz[2] has found that the index increases from 2 to 5 and then decreases to 1.55, as the speed of the projectile is increased up to and beyond the velocity of sound in air.

In aërodynamics the investigation of the laws of air resistance for large bodies moving at relatively high speeds has been for several years one of the chief occupations of aëronautical research. Zahm,[3] Lanchester[4] and others have studied the problem and find two factors, one called the head resistance, which varies with the square of the velocity within certain limits, and a second form of resistance, caused by surface friction, which varies with the 1.86 power of the velocity. It is of interest to electrical engineers to know that formerly some of the electrical supply meters depended for their operation upon the motion of fans in air or liquids, which caused resisting forces varying with the square of the speed. Such were the Forbes, the Schallenberger, the Ferranti and the Slattery meters.[5] A close similarity exists between the action of the retarding forces in air and in water. It has been shown experimentally that the exponent for surface friction when water flows through pipes and channels is 1.85; this is almost exactly the value given by Zahm for the resistance of the air when flat or tapering bodies move edgewise. If the flow of the water is turbulent the index rises to the square,[6] corresponding thus to the expression for the resistance met by blunt bodies propelled through air. In viscous fluids, Stokes's law of resistance varying with the first

[1] Sir Isaac Newton, Principia, Book II., Sec. VII.

[2] Cranz, "Ballistik," Encykl. der math. Wissenschaften, Vol. IV., Part 2, 1903. Contains a complete bibliography. See also Encycl. Brit., Ballistics.

[3] Zahm, Philosoph. Soc. Washington, 1904.

[4] Lanchester, Aërial Flight, Vol. I., Chap. 2, 1908; A discussion concerning the Theory of Sustentation and Expenditure of Power in Flight, Eng. Cong., San Francisco, 1915; Berriman, The "Arrival" of the Aëroplane, Eng. Cong., San Francisco, 1915.

[5] Swinburne, The Measurement of Electrical Currents, 1893; Parr, Electrical Engineering Measuring Instruments, 1903.

[6] Knibbs, Proc. Roy. Soc. N.S.W., Vol. 31, 1897.

power of the velocity, based on the assumption of non-sinuous motion with no slipping at the boundary, was found by Allen[1] to hold for the motion of bubbles and small solid spheres in liquids if the velocity remained small, but if the velocity was increased sufficiently the resistance approximated the square law closely.

An interesting problem in hydraulic engineering appears in the design of surge towers to regulate the oscillations occurring in long conduits when the rate of flow is suddenly varied. If the flow is shut off entirely, the surges arising between the tower and the adjoining conduit represent in effect a freely oscillating system damped by resistance approximately proportional to the square of the velocity. In most of the papers listed below[2] a graphical or step-by-step method of integration is presented; in some the resistance is assumed to vary with the first power of the velocity; in none is a strictly analytical solution given for the oscillations when the resistance is assumed proportional to the square of the velocity. The resistance to the motion of vessels has been found to be similar to that discussed in the paragraph on aërodynamics. The head resistance varies with the square of the speed while the frictional resistance is proportional to a somewhat lower index. For the case of a ship rolling in still water White[3] assumed the resistance to vary partly as the angular velocity and partly as the square of the angular velocity. When rolling among waves the motion of the ship becomes a forced oscillation with a periodic forcing cause and damped by resistance varying with both the first and second powers of the angular velocity.[4]

This brief discussion is in no sense complete; it is intended merely to suggest the wide scope and prevalence of the square law of damping, and to indicate certain problems involving oscillating systems which are presented for solution.

II. Free Oscillations Damped by Resistance Proportional to the Square of the Velocity.

The equation of motion for a system executing free oscillations and opposed only by a force varying as the square of the velocity is

[1] Allen, Phil. Mag., Ser. 5, Vol. 50, 1900, pp. 323 and 519.

[2] Johnson, The Surge Tank in Water Power Plants, Trans. Am. Soc. Mech. Eng., Vol. 30, p. 443, 1908; Uhl, Speed Regulation in Hydro-Electric Plants, ibid., Vol. 34, p. 379; 1912; Durand, On the Control of Surges in Water Conduits, ibid., Vol. 34, p. 319, 1912; Warren, Penstock and Surge Tank Problems, Proc. Am. Soc. Civ. Eng., Vol. 40, 2, p. 2521, 1914; Church, Surge in a Hydraulic Stand Pipe, Cornell Civ. Eng., Dec., 1911; Forcheimer, Zeitschr. d. Ver. Deutsch. Ing., Vol. 56, p. 1291, 1912 (includes a bibliography of the foreign literature); Prasil, Surge Tank Problems, The Canadian Eng., 1914 (includes a bibliography).

[3] Sir Wm. H. White, Trans. Inst. Nav. Arch., 1895.

[4] Sir Philip Watts, Shipbuilding, Ency. Brit., 1910. See also Henderson, Engineering, April 18, 1913.

$$\frac{d^2\theta}{dt^2} \pm N\left(\frac{d\theta}{dt}\right)^2 + M\theta = 0. \tag{1}$$

Poisson[1] and others have investigated this equation for small oscillations, considering the higher powers of the successive arcs as negligible. Routh[2] performs the first integration and obtains a relation between the velocity and the displacement. Grammel[3] and Ignatowsky[4] give approximate solutions showing the displacement in terms of the time, useful when the oscillations are small but not adapted to larger values of the displacement. In the following pages a solution is developed which is particularly adapted to large displacements and heavy damping. In addition an experimental method is presented, by which it has been found possible to verify the results of the analytical study.

Transforming equation (1) by means of the proper substitutions and performing the first integration, the expression for the velocity in terms of the displacement is

$$\left(\frac{d\theta}{dt}\right)^2 = \frac{M}{2N^2} \mp \frac{M}{N}\theta + C_1\epsilon^{\mp 2N\theta}, \tag{2}$$

wherein C_1 is the constant of integration and depends on initial conditions. In order to define the meaning of the remaining symbols let us consider the motion of a torsion pendulum to be thus expressed. Then θ is the angle of displacement from the position of equilibrium; M is the restoring torque in cm.-dynes per radian of twist, divided by the moment of inertia of the pendulum about the axis of support; N is the coefficient of resistance, that is, the torque per square of unit angular velocity divided by the moment of inertia. The double sign before N is necessary because a change in the direction of the motion does not automatically change the sign of the resistance term. At each turning point, therefore, a discontinuity is introduced into the equation.

The integral of equation (2) expresses the relation between the angle of displacement θ and the time T.

$$\int dt = \pm \int \frac{d\theta}{\left(\frac{M}{2N^2} \mp \frac{M}{N}\theta + C_1\epsilon^{\mp 2N\theta}\right)^{1/2}}. \tag{3}$$

It is not possible to integrate the right-hand member of equation (3) in the form as written. For certain restricted values of N and θ, however, it is possible to express this integral in terms of a converging series and

[1] S. D. Poisson, A Treatise of Mechanics, Vol. I., Sec. 186–190.

[2] E. J. Routh, Dynamics of a Particle, Art. 129, 1898.

[3] R. Grammel, Physik. Zeitschr., Vol. 14, p. 20, 1913.

[4] W. v. Ignatowsky, Archiv. d. Math. u. Phys., Vol. 17, p. 338, 1910.

thus to obtain a relation between θ and the time. Substituting $z = (1 \mp 2N\theta)$ and assuming as initial conditions $\theta = \theta_0$ and $d\theta/dt = 0$ when $T = 0$, in order to determine C_1, equation (3) becomes

$$T - C = -\frac{1}{\sqrt{2M}}\int [z]^{-1/2}\left[1 - \left(\frac{z_0}{e^{z_0}}\right)\frac{\epsilon^z}{z}\right]^{-1/2} dz. \tag{4}$$

C is the constant resulting from the second integration and z_0 is the value of z at $T = 0$. The upper sign in the expressed value of z corresponds with positive angular velocity, the lower sign with negative angular velocity. If the absolute value of $(z_0\epsilon^z/\epsilon^{z_0}z)$ is less than unity, the right-hand member of equation (4) may be expanded according to the binomial theorem into an infinite series which converges absolutely. Assuming for the moment that the values of N and θ are such that this expansion is permissible,

$$T - C = -\sqrt{\frac{2}{M}}\int\left[1 + \tfrac{1}{2}k\frac{\epsilon^{y^2}}{y^2} + \tfrac{3}{8}k^2\frac{\epsilon^{2y^2}}{y^4} + \tfrac{5}{16}k^3\frac{\epsilon^{3y^2}}{y^6} + \cdots \right.$$
$$\left. \cdots + \frac{(-\frac{1}{2})(-\frac{1}{2}-1)\cdots(-\frac{1}{2}-n+1)}{n!}(-k)^n\frac{\epsilon^{ny^2}}{y^{2n}} + \cdots\right]dy, \tag{5}$$

where $k = z_0/\epsilon^{z_0}$ and $z = y^2$. The integration of the series, equation (5), may be expressed as the sum of the integrals taken term by term. The general integral is

$$a_n\int\frac{\epsilon^{ny^2}}{y^{2n}}dy = a_n\int\frac{1}{y^{2n}}\left[1 + ny^2 + \frac{n^2y^4}{2!} + \frac{n^3y^6}{3!} + \cdots\right]dy. \tag{6}$$

For values of $z_0\epsilon^z/\epsilon^{z_0}z$ less than 0.99 it is sufficient for most purposes to consider the first six terms of the binomial expansion, equation (5), and the first seven terms of the expansion of each integral. The result is the following equation expressing the time T in terms of the displacement θ.

$$T - C = -\sqrt{\frac{2}{M}}\left[\left((1 + \alpha)y + \frac{\beta}{3}y^3 + \frac{\gamma}{5}y^5 + \frac{\delta}{7}y^7 + \frac{\zeta}{9}y^9 + \frac{\eta}{11}y^{11}\right)\right.$$
$$\left. - \left(\frac{\alpha}{y} + \frac{\lambda}{3y^3} + \frac{\mu}{5y^5} + \frac{\nu}{7y^7} + \frac{\rho}{9y^9}\right)\right], \tag{7}$$

where $y = \sqrt{z} = \sqrt{1 \mp 2N\theta}$. The constants $(\alpha, \beta, \gamma, \delta, \zeta, \cdots)$ are found by collecting the coefficients of the terms in like powers of y obtained from the expansion of each member of equation (5). A simple relation connects each constant with the preceding constant. All of the terms in the right-hand member of (7) are known when the constants N and M are specified and the desired value of θ is chosen. We have therefore a complete solution of the differential equation (1), subject only to the limitation in the values of N and θ necessary to make the expansion absolutely convergent.

If $z_0\epsilon^z/\epsilon^{z_0}z$ is greater than unity equation (4) may be written

$$T - C = -\frac{1}{\sqrt{2M}}\int [-k\epsilon^z]^{-1/2}\left[1 - \left(\frac{\epsilon^{z_0}}{z_0}\frac{z}{\epsilon^z}\right)\right]^{-1/2} dz. \tag{8}$$

This expresses the product of two absolutely convergent series. It may therefore be developed by the method outlined above and a second solution for T found, in terms of the displacement θ. This solution will be applicable for all values of N and θ for which (7) is not valid, save for the unique case $z_0\epsilon^z/\epsilon^{z_0}z = +1$ which renders both forms of solution indeterminate. It will be noted that at $T = 0$, $z = z_0$ and the solution assumes the indeterminate form. Hence it is necessary to know the value of T for any one angle θ other than θ_0, say the value of T when $\theta = 0$ and the vibrating system is passing through the position of equilibrium. At this value the series converges very rapidly. The constant of integration may be derived as well from the value of T at the first turning point, but here the expression (7) is in general less rapidly convergent.

We have in equations (7) and (8) relations by which the value of the displacement may be found at any instant. It is often desirable and sufficient for the purpose at hand to find merely the value of θ at the turning points. This may be done by placing $d\theta/dt = 0$ in equation (2) and determining C_1 from the condition that $\theta = \theta_0$ when $T = 0$. Then if θ_1 is the displacement at the first turning point,

$$(1 \mp 2N\theta_1)\epsilon^{\pm 2N\theta_1} = (1 \mp 2N\theta_0)\epsilon^{\pm 2N\theta_0} \equiv A. \tag{9}$$

Expanding $\epsilon^{\pm 2N\theta_1}$ into an infinite series which converges absolutely for all values of $\pm 2N\theta_1$, equation (9) may be reduced to the following form:

$$2N\theta_1 = \sqrt{\frac{1 - A}{.5 + X}}, \tag{10}$$

where

$$X = \frac{2N/\theta_1/}{3} + \frac{(2N\theta_1)^2}{8} + \frac{(2N/\theta_1/)^3}{30} + \frac{(2N\theta_1)^4}{144} + \cdots. \tag{11}$$

All of the terms in X are additive, even when the angle θ_1 is negative; it is a rapidly converging series, especially when $2N/\theta_1/$ is less than unity. In general $2N/\theta_1/$ is considerably less than unity so that the equation presents a practical form for finding θ_1 by successive approximations. It appears also from equation (10) that the ratio of successive arcs is independent of the magnitude of the restoring torque M, contrary to the relations obtaining when the damping is proportional to the first power of the velocity.

In order to study the period of the oscillations return to equation (2). If V_0 is the velocity of the moving system when $\theta = 0$ then $V_0^2 = M/2N^2 + C_1$ and equation (2) may be written:

$$V^2 = \left(\frac{d\theta}{dt}\right)^2 = V_0^2 \mp 2V_0^2N\theta + 2C_1N^2\theta^2[1 \mp \tfrac{2}{3}N\theta + \tfrac{1}{3}N^2\theta^2 \mp \tfrac{2}{15}N^3\theta^3 + \cdots]. \qquad (12)$$

If the product $N\theta$ is considerably less than one the expression within the brackets will not differ much from unity. We may therefore write

$$T - C = \int [V_0^2 \mp 2V_0^2N\theta + 2C_1N^2\theta^2]^{-1/2}d\theta = \sqrt{\frac{\epsilon^{(z_0-1)}}{Mz_0}}\sin^{-1}\left(\frac{-2C_1N^2\theta \pm NV_0^2}{\sqrt{N^2V_0^4 - 2C_1N^2V_0^2}}\right), \qquad (13)$$

from which

$$\theta = -\frac{\sqrt{V_0^4 - 2C_1V_0^2}}{2C_1N}\sin\left[T\sqrt{\frac{Mz_0}{\epsilon^{(z_0-1)}}} - C'\right] \pm \frac{V_0^2}{2C_1N}. \qquad (14)$$

If T_0 is the period of the oscillation then

$$T_0 = \frac{2\pi}{\sqrt{M}}\sqrt{\frac{\epsilon^{+2N/\theta_0/}}{(1 + 2N/\theta_0/)}}. \qquad (51)$$

When N, the coefficient of damping, is zero $T_0 = 2\pi/\sqrt{M}$, which is the usual expression for the period of undamped harmonic vibrations. If $2N/\theta_0/ = 0.10$ the period is increased by 0.25 per cent. above that for undamped motion. This indicates that for moderate values of damping and displacement the period is lengthened slightly and the motion remains practically isochronous. If $2N\theta_0$ approaches a value of unity or greater, equation (15) is meaningless because the approximation employed is no longer valid. It may be shown experimentally, however, that for the larger values of $2N\theta_0$ the period is a little longer and the motion is still almost isochronous, just as in the case where the resistance varies as the first power of the velocity.

If values of θ are known at the successive turning points of the motion, by observation for example, an approximate value of N, the coefficient of resistance, may be found by a consideration of the energy relations during any one oscillation. At the initial position the velocity is zero and all of the energy is potential; this is true also at the next point of rest θ_1. The difference between the amounts of potential energy is equal to the dissipation of energy caused by the resistance, during the excursion from θ_0 to θ_1. Referring again to the torsion pendulum, if τ is the static moment of force per radian of twist in the suspending wire;

I_c is the moment of inertia, and N is the coefficient of resistance as before,

$$\frac{\tau}{2}[\theta_0^2 - \theta_1^2] = -NI_c\int_{\theta_0}^{\theta_1}\left(\frac{d\theta}{dt}\right)^2 d\theta. \tag{16}$$

If $d\theta/dt$ can be expressed as an integrable function of θ, this equation will serve for the determination of N. Let the velocity be plotted in terms of displacement (Fig. 1) from equation (2), assuming for the moment any reasonable value of N. For moderate values of damping this curve during any one oscillation suggests a portion of an ellipse, with its center displaced from the origin of coördinates to the point corresponding to maximum velocity. That is, the motion is approximately simple harmonic about a center which is moved first to one side, then to the other side of the origin. Let θ' be the value of θ when the velocity is a maximum, and write $\alpha = \theta - \theta'$, then

$$\left(\frac{d\alpha}{dt}\right)^2 = \left(\frac{d\theta}{dt}\right)^2 = \frac{\tau}{I_c}[(\theta_0 - \theta')^2 - (\theta - \theta')^2] \tag{17}$$

expresses the simple harmonic relation between velocity and displacement. Substituting in equation (16), integrating and solving for N, its approximate value is

$$N = 3\frac{[\theta_0^2 - \theta_1^2]}{[\theta_0 + /\theta_1/]^3}. \tag{18}$$

This approximate value may then be substituted in equation (10) and by successive trials a new value of N may be found which will correspond with the observed turning points, with any required degree of precision.

As a check upon the method of solution elaborated in the preceding pages, it may be noted that equation (2) can be integrated by graphical means. For example, let $N = 1.0$; $M = 10.0$; $\theta_0 = 1.0$ radian. If θ be taken as the independent variable and the reciprocal of the velocity be plotted against θ, the curve shown in Fig. 1 is obtained. Discon-

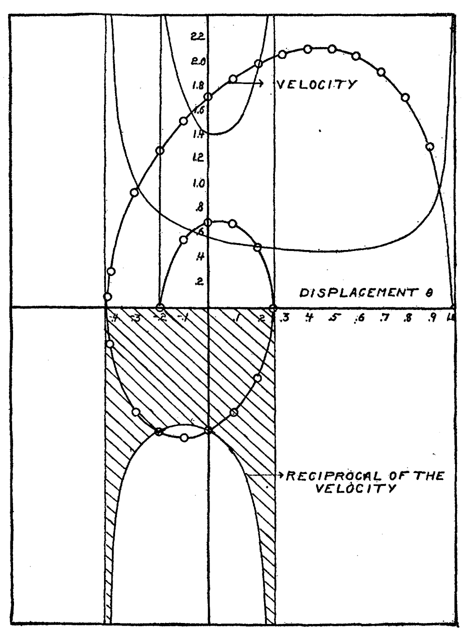

Fig. 1.

tinuity of resistance requires the use of the double sign in equation (2), and the constant of integration must be redetermined for each successive excursion. The area of any narrow strip extending from this curve to the axis of θ represents the amount by which T is increased, while the moving system passes through the corresponding angle. Using a planimeter, T may then be obtained in terms of θ by means of a step by step area summation. Plotting θ in terms of T, the curve shown in Fig. 2 is a graph of the motion demanded by the original differential equation.

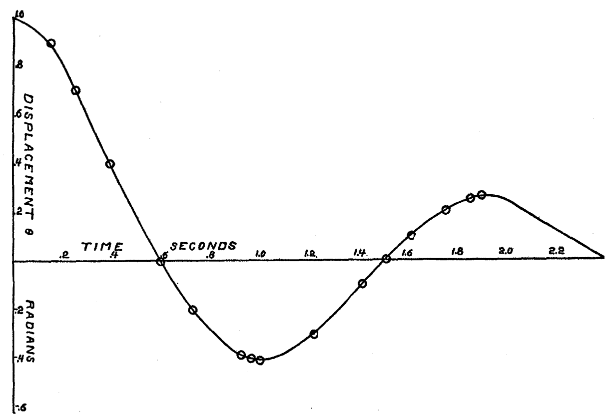

Fig. 2.

Plot of $\frac{d^2\theta}{dt^2} \mp 1.0 \left(\frac{d\theta}{dt}\right)^2 + 10.0\,\theta = 0$

Full Line by Fig. 1. Circles by equation (7).

To compare with the results of the graphical integration, the same values of N, M and θ_0 were substituted in equation (7). T was thus obtained for several values of θ. The results are shown as small circles placed upon the curve, Fig. 2. The agreement between graphical integration and the computations from equation (7) is apparent.

III. Experiments.

Few attempts have been made to construct models for the study of oscillations damped by resistance varying with the square of the velocity. Durand[1] has suggested the use of small models of surge chambers and the extension of the results of observations, by the law of kinematic similitude, to surge towers of full size. No other reference has been found to any experimental means of reproducing and studying oscillations of this kind. It is obvious that any type of pendulum may be used, provided that the resistance is convertible into a function of the square of the velocity. It

[1] W. F. Durand, Trans. Amer. Soc. Mech. Eng., Vol. 34, p. 359, 1912.

will be advantageous to choose a model such that the coefficients of the resisting and the restoring forces can be varied at will, in which a minimum of inherent errors and unknown factors is introduced, and for which the required velocity is not so great as to render accurate observation difficult. There are doubtless many ways of obtaining an automatic adjustment of the resistance. Certain inherent difficulties appear to offset the advantages of automatic control; in the following pages a method depending essentially on electromagnetic induction is described, in which the necessary simplification is obtained by means of a manually operated device.

An annealed copper disk, 15 cm. in radius, is suspended from a rigid support by a phosphor-bronze torsion wire. The disk is accurately leveled and centered and oscillates between the jaws of a small soft iron electromagnet in such a manner that eddy currents are generated in the disk. The electromagnet is hung by a silk thread and hence is free to respond to the drag of the rotating disk caused by the Foucault currents. Attached to the lower jaw of the magnet is a small spiral spring which resists the tendency to turn, so that the actual displacement of the electromagnet from its position of equilibrium depends directly on the speed of rotation of the disk. The image of a brightly lighted, vertical slit is reflected from a mirror attached to the back of the magnet to a volt-potentiometer arranged in the form of an arc. The distance of the spot of light from the center of the arc thus gives a continuous measure of the velocity of the rotating disk.

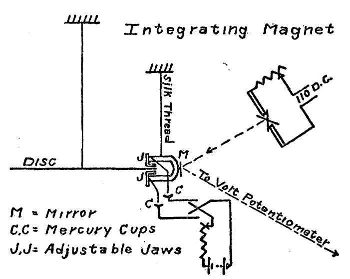

Fig. 3.

The volt-potentiometer consists of a meter stick bent into an arc of radius 55 cm. and effective length 60 cm. Every half centimeter there is a commutator segment consisting of a copper wire held firmly against the face of the stick. Each pair of segments equidistant from the center is connected through a fuse and switchboard to a supply of four volts from a large lead cell storage battery. Completing the circuit between the potentiometer center and the sliding contact are two pairs of specially designed damping coils, in the field of which the copper disk oscillates. Each of the coils has a radius of 6 cm., a wooden core of 2 cm. radius and is wound with 52 layers of number 20 enameled copper wire. The air gap in which the disk rotates is just large enough to permit perfect clearance. It is thus possible to obtain very heavy damping. The sliding

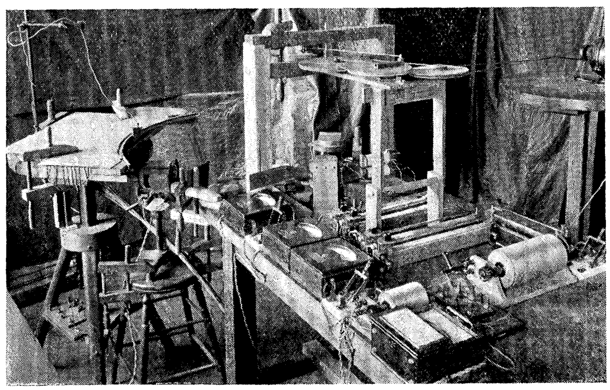

Fig. 5.

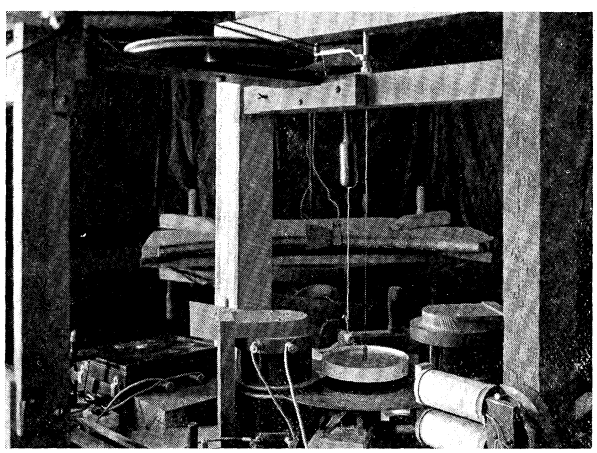

Fig. 6.

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contact on the volt-potentiometer is kept by hand under the spot of light reflected from the mirror on the integrating magnet. Hence at any instant the voltage across the damping coils is directly proportional to the velocity of the moving disk. Since there is no iron present and the speed of the disk is moderate, the current through the damping coils will vary sensibly as the voltage applied. Therefore the field set up by the coils varies directly as the velocity of the disk. Now the eddy currents generated in the rotating disk are directly proportional to the strength of the field, and the torque developed, which opposes the motion, is directly proportional to the product of the generated eddy currents and the existing field. Therefore the oscillations of the torsion pendulum are damped by a resistance varying as the square of the velocity. To regulate the value of the coefficient of damping it is only necessary to change the maximum voltage applied across the coils; or the deflection of the spot of light may be varied as desired, by changing the current actuating the electromagnet.

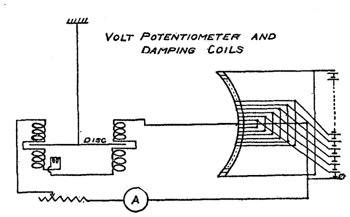

Fig. 4.

For the purposes of observation a circular millimeter scale 40 cm. in length is placed on the disk and turns with it. A fixed pointer leading out to the scale enables the successive turning points to be read with an accuracy of one part in four hundred, provided that the period of the swing is moderately large. If a continuous record of the displacement is desired a photographic record of the rotating scale may be taken. When the torsion wire is initially twisted, a device beneath the disk clamps the axle in such a manner that the disk may be released without jar. The torsion wire is joined to the disk by a small cap which is screwed on the axle, so that a new suspension may be substituted easily. The model may readily be adapted to show forced oscillations, motion damped by a combination of first power and second power resistance, rectilinear damping, and so on. (See Figs. 5–6.)

Considering the inherent errors of the apparatus: (1) There is always present a small amount of damping proportional to the first power of the speed. This is due to the drag of the integrating magnet and to the slight air friction on the upper and lower surfaces of the thin copper disk. (2) The current flowing in the coils and hence the field set up, is not exactly proportional to the voltage applied because the electromotive force is not constant but varying. In all of the measurements the resistance of this circuit in ohms was at least seventy times the inductance of the damping coils, measured in henrys, and the period of oscillation of the disk was always greater than ten seconds. It may easily be shown by integration of the equation of electromotive forces that no appreciable error is introduced when the current is assumed proportional to the applied voltage. (3) An effect necessary to overcome was the influence of the varying field of the damping coils on the suspended magnet. To obviate this, a small coil was placed between the damping coils and the magnet and connected in series with the coils in such a way that its field almost entirely neutralized the troublesome stray induction. It was also found advantageous to maintain a weak general field about the magnet in order to reduce the importance of stray magnetic fields.

The copper disk weighing 755.5 grams was suspended by a No. 22 phosphor-bronze torsion wire, 35 cm. in length. From the change in the period due to the addition of a turned brass cylindrical disk weighing 976.5 gm. and also by direct computation from dimensions, the moment of inertia about the axis of support and the constant of the torsion wire were found; by means of these values the coefficient of the restoring torque, M, was found to equal 0.294 c.g.s. units. The natural, or undamped period, measured by chronograph and stop-watch, was 11.604 seconds. A current of 0.60 amperes was then sent through the electro-

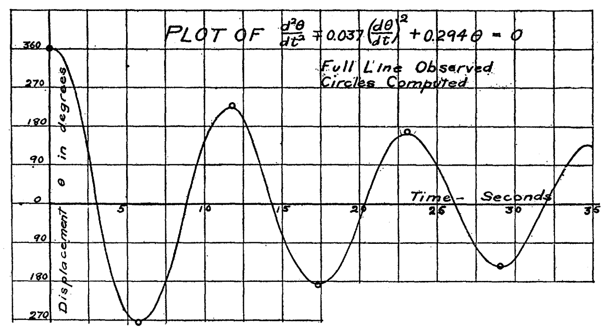

Fig. 7.

magnet circuit, a maximum voltage of 216 volts was applied across the potentiometer, the disk was displaced initially through 360 degrees, and turning points were observed. The motion is plotted in Fig. 7. The unknown coefficient of damping was found approximately, $N = 0.0370$. This value of N was used to determine from equation (10) the values of successive turning points, under the assumption of a resistance varying solely with the square of the velocity. The values are shown in Fig. 7 by circles. The effect of the small unavoidable damping, varying with the first power of the velocity, is apparent, the first, second, third and fourth observed turning points lying a little inside of the computed values. It was found that the greatest variation from pure square damping between any two consecutive oscillations was 2.5 per cent. This is within the anticipated minimum of unavoidable error.

The next step was a more detailed comparison of the analytical solution, equation (7), with an experimental system. The restoring torque was maintained at its former value, but the coefficient of damping was increased by means of larger current in the damping coils. From several observations the first three turning points were found to be 360°, 197°, 149°. The new coefficient of damping was computed approximately from equation (18) and more precisely from equation (10). $N = 0.0960$. The constant of integration for equation (7) may be computed if we know the period of the oscillation. Experimentally the half-period, or the time for the first swing, was found to be 5.860 seconds. It may also be determined analytically from the relation

$$T\sqrt{M} = T_1\sqrt{M_1}(K), \tag{19}$$

where T, M and T_1, M_1 are the half-periods and coefficients of the restoring torques, respectively, for two independent systems and (K) is the ratio of their correction factors (see equation (15)). Let us take T_1, M_1 for the case solved by graphical integration, Fig. 2; then $T = 5.840$ seconds, which is in close agreement with the experimental determination. All of the terms in equation (7) are now known. It may therefore be applied to the first oscillation in the manner already outlined. The graph of the motion is plotted in Fig. 8 and the values as computed by equation (7) are indicated by circles. It will be seen that the agreement is entirely satisfactory. For the second half-period, equation (7) is not applicable because the original expansion is not convergent. To determine the motion for the second oscillation analytically, it would be necessary to employ the expanded form resulting from the development of equation (8).

Figs. 7 and 8 exhibit the motion of the oscillating systems as practically isochronous. We have proved analytically that this is true for moderate damping. (See equation (15).) It may also be shown experimentally. A smoked drum chronograph 50.5 cm. in circumference was driven by a small motor connected through a friction drive and reduction gears. Two needle pointers mounted on a traveling screw traced continuous paths on the drum. One pointer was actuated by a contact on the pendulum of a laboratory clock so that it indicated seconds. The other pointer, by means of a tapping key, was made to record the successive instants at which the torsion pendulum passed through its position of equilibrium. For the motion of the oscillating system plotted in Fig. 7 five independent records of the times for successive swings were taken and the first nine average values were found to be 5.81, 5.85, 5.77, 5.87, 5.77, 5.83, 5.78, 5.81, 5.82 seconds. The average half-period was a little more than 5.81 seconds and the greatest deviation from the average was

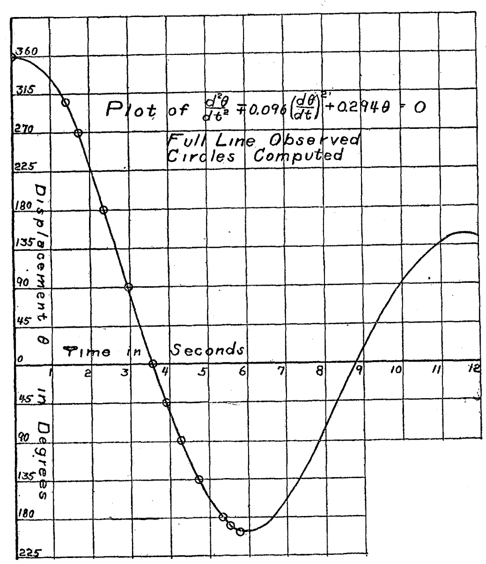

Fig. 8.

0.06 second, or approximately one per cent. This is well within the expected accuracy, since it is probable that for any single observation an error of one tenth of a second would easily enter in tapping the key a little too soon or too late. We are therefore furnished with experimental evidence that for moderate and even heavy damping the oscillations are practically isochronous.

IV. Use of the Experimental Model in the Study of Other Forms of Damped Oscillations.

It has been mentioned that the model described in Part III. is capable of extension to other problems in damped harmonic motion. Of these problems, three have been given some consideration, experimentally and analytically. These are: (1) free oscillations subject to both first and second power damping, the first power predominating; (2) forced, continuous vibrations; (3) rectilinear damping. A detailed discussion of each case is beyond the scope of this paper. We shall merely point out the lines along which the investigation has proceeded.

1. In order to produce a resisting torque due to both first and second power damping it is only necessary to add to the existing arrangement a constant damping field. By means of a small coil with an iron core (demagnetized before each setting), through the field of which the disk rotates, a wide range of damping is readily obtained. (See Fig. 6.) One feature of the greatest advantage lies in the ability to study each of the two components of the damping singly as well as combined. We may therefore make an independent determination of each coefficient of damping apart from the more complicated motion due to their joint action. Analytically the oscillations may be treated by the double approximation method of Routh,[1] provided that the first power damping predominates. The following conclusions have been established experimentally for systems having an initial displacement of 2π radians: (*a*) the oscillations are isochronous; (*b*) the period is increased somewhat by an increase in either or both components of the damping; (*c*) the ratio of arcs is at first greater and then less than the average value; (*d*) successive differences for the combined damping are at first greater and then less than for the first power damping alone; (*e*) the influence of the second power damping rapidly diminishes and the decrement approaches its value for pure first power resistance. This is to be expected since a decrease in the velocity to one half its original value reduces the second power damping to one fourth its initial effectiveness.

2. Forced oscillations may be produced by twisting the torsion wire periodically. To accomplish this, an arm is rigidly attached to the torsion wire head and is driven back and forth harmonically by means of a connecting rod, slotted cross head and reduction gears. (See Fig. 5.) If higher harmonics are desired in the forcing term, a combination of gears such as are used in a curve-tracer may be employed to turn the torsion wire. Following Routh's double approximation method a solution may be developed for systems in which the amount of damping is small.

3. The influence of a constant damping factor, independent of the speed, on the motion of an oscillating system becomes of importance in several interesting connections. Thus from the equations derived by Durand[2] for the surge-chamber problem in hydraulics, if the effect of governor control is neglected and the motion of the water is expressed solely in terms of displacement in the surge tower, there results an equation of the type

[1] E. J. Routh, Adanced Rigid Dynamics, p. 251, 1905. The solution given by Routh (Art. 364) is limited to displacements so small that their squares and higher powers may be neglected. It is possible, however, following Routh's general method to develop a solution not limited by the magnitude of the displacement.

[1] W. F. Durand, *op. cit.*, p. 321, 1912.

$$\frac{d^2\theta}{dt^2} \pm A\left(\frac{d\theta}{dt}\right)^2 + B\left(\frac{d\theta}{dt}\right) + C\theta + D = 0. \qquad (20)$$

Here there are present three types of damping: that varying with the first power of the speed; that varying with the second power, and a constant resistance independent of the speed. The model which has been developed may be adapted to exhibit just such a type of resisted motion.

In connection with electrical instruments subject to pivot friction and in certain oscillating systems, Blondel and Carbenay[1] have shown that the oscillations are represented by an equation of the type

$$\frac{d^2\theta}{dt^2} + B\left(\frac{d\theta}{dt}\right) + C\theta + D = 0 \quad (\text{or} = A \sin(\omega t)). \qquad (21)$$

An electrical analogy is found in an oscillating circuit of radio frequency in which spark resistance predominates.[2] An experimental study of motion corresponding to the type equation (21) may readily be made. It is beyond the limits of this paper, however, to enter into a discussion of these allied problems in resisted oscillations. Enough has been suggested, perhaps, to indicate the flexibility of the model developed and the wide range of problems for which there is now a means of experimental study.

Summary.

1. A complete solution has been developed for the equation

$$\frac{d^2\theta}{dt^2} \pm N\left(\frac{d\theta}{dt}\right)^2 + M\theta = 0,$$

subject only to one limitation, namely:

$$\frac{(1 \pm 2N\theta_0)\epsilon^{1\pm 2N\theta}}{\epsilon^{1\pm 2N\theta_0}(1 \pm 2N\theta)} \neq 1.$$

2. The solution has been verified by comparison with the results of graphical integration and by experiment.

3. The oscillations have been shown analytically and experimentally to be practically isochronous.

4. A solution has been developed by which the numerical values of the successive turning points may be computed.

5. A model has been constructed suitable for a wide range of systems by means of which it is possible to study oscillations damped by resistance proportional to the square of the velocity.

[1] A. Blondel and F. Carbenay, La Lumière Élec., Nov. 27, Dec. 4, 11, 1915, and July 29, 1916.

[2] A. W. Stone, Inst. Radio. Eng., Proc. 2, pp. 307–427, Dec., 1914.

6. A method is given for the determination of the coefficient of damping in terms of the values of the successive turning points.

7. The model developed may be extended to exhibit: (*a*) forced oscillations; (*b*) systems resisted by both first and second power damping; (*c*) systems resisted by a constant factor, alone or in combination with other forms of damping.

8. The prevalence of damping proportional to the square of the speed in problems of resisted motion has been demonstrated by a survey of the literature. A bibliography is given for the more important engineering cases in which this type of oscillation occurs.

In conclusion, the author takes pleasure in expressing his thanks to Prof. W. J. Raymond, Prof. B. M. Woods, Prof. F. E. Pernot and others, whose valuable suggestions and friendly criticisms have been most helpful.

Physical Laboratory, University of California,
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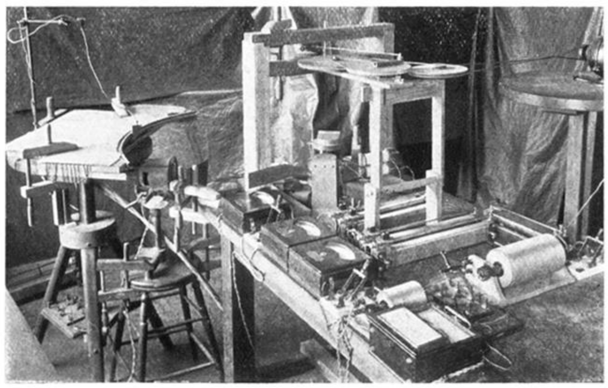

Fig 5.

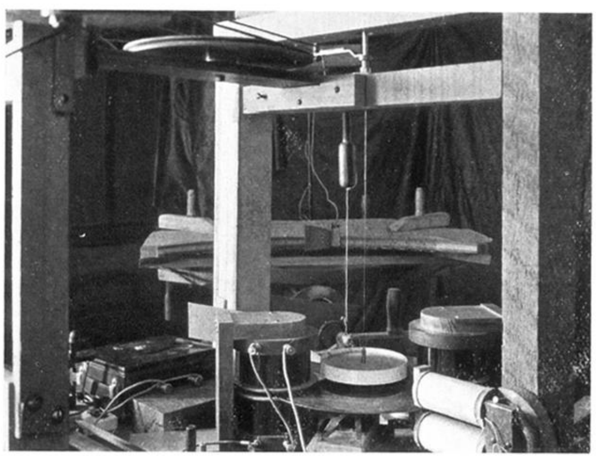

Fig. 6.